p-adic Interpolation of Rankin-Selberg convolution

Master's thesis report

Sanyam Gupta

Under the supervision of Mladen Dimitrov

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University of Lille

France

Contents

CHAPTER 1

General overview

Let p be a prime number. In this thesis, we present Hida's method for constructing a p-adic L-function that p-adically interpolates the *special values of the Rankin-Selberg convolution* of two elliptic modular forms of unequal weights. Roughly speaking, a p-adic L-function is a "p-adic analytic" function whose values coincide with those of its complex analytic counterpart at sufficiently many integer points to ensure its uniqueness.

We achieve the desired p-adic L-function by first constructing a measure with values in the space of modular forms, starting from naturally defined distributions with values in this space (such as Eisenstein distributions, theta distributions, etc.). To obtain a numerically valued distribution from these distributions with values in the space of modular forms, we apply a suitable linear form derived from the Rankin-Selberg method.

However, before delving into the specifics of this construction, we take the opportunity to provide a general overview and some motivation for studying p-adic L-functions.

1. Classical L-functions

The study of L-functions and their special values dates back centuries and continues to play a fundamental role in modern number theory. Roughly speaking, they are Dirichlet series $\sum a_n n^{-s}$ whose coefficients a_n contain arithmetic information. Below, we describe some of the most important examples of L-functions.

Example 1.1 (Riemann zeta function). The most famous example is the Riemann zeta function:

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s} = \prod_p (1 - p^{-s})^{-1}, \quad s \in \mathbb{C},$$

where the product runs over all prime numbers p. The second equality is a consequence of the *unique factorization theorem.* The sum converges absolutely for the real part of s greater than 1, making ζ a holomorphic function in a right half-plane. The expression as a product is called an *Euler product.* Furthermore, it is well known that for

$k \in \mathbb{Z}_{>0}$ we have:

$$\zeta(1-k) = -\frac{B_k}{k},$$

where B_k denote the n-th Bernoulli number. The Bernoulli numbers are combinatorial in nature and satisfy some recurrence relations that ensures they are rational numbers. Thus, the values of ζ function at negative integers are rational numbers: $\zeta(1-k) \in \mathbb{Q}$ for $k \in \mathbb{Z}_{>0}$, and $\zeta(1-k) = 0$ if $k > 2$ is odd.

n	0	1	2	3	4	5
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0

TABLE 1. First six Bernoulli numbers

EXAMPLE 1.2 (Dedekind zeta function). Another important example is the Dedekind zeta function attached to a number field. Let F be a number field. The zeta function ζ_F of F is

$$\zeta_F(s) := \sum_{0 \neq I \subset \mathcal{O}_F} N(I)^{-s} = \prod_{\mathfrak{p}} (1 - N(\mathfrak{p})^{-s})^{-1}, \quad s \in \mathbb{C},$$

where the sum is over the non-zero integral ideals of F. This sum again converges for the real part of s greater than 1, making ζ_F a holomorphic function in the right half-plane $\mathrm{Re}(s) > 1$. The product runs over all the prime ideals of F. This is again called the Euler product, and its existence follows from the unique factorization of ideals in the ring of integers $\mathcal{O}_F$.

EXAMPLE 1.3 (Dirichlet L-functions). Let $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character, and extend it to a function $\chi : \mathbb{Z} \to \mathbb{C}$ by setting it to be 0 at integers not coprime to N. The L-function of χ is

$$L(\chi, s) = \sum_{n=1}^{\infty} \chi(n) n^{-s} = \prod_p (1 - \chi(p) p^{-s})^{-1},$$

which again converges in a right half-plane.

EXAMPLE 1.4 (L-function of an elliptic curve). Let $E/\mathbb{Q}$ be an elliptic curve of conductor N. The associated L-function is defined as

$$L(E, s) = \sum_{n \geq 1} a_n(E) n^{-s} = \prod_{p \nmid N} (1 - a_p(E) p^{-s} + p^{1-2s})^{-1} \prod_{p \mid N} L_p(s),$$

where $a_p(E) = p + 1 - \#E(\mathbb{F}_p)$, and the factors $L_p(s)$ at bad primes are defined as $L_p(s) = 1$ (resp. $(1 - p^{-s})^{-1}$, resp. $(1 + p^{-s})^{-1}$) if E has bad additive (resp. split multiplicative, resp. non-split multiplicative) reduction at p.

EXAMPLE 1.5 (L-function of a modular form). Let $f = \sum_{n=1}^{\infty} a_n(f)q^n \in S_k(\Gamma_0(N), \omega_f)$ be a (normalized) newform of weight k, level N, and character ω_f. The L-function associated to f is given by

$$\begin{aligned} L(f,s) &= \sum_{n\geq 1} a_n(f)n^{-s} \\ &= \prod_{p\nmid N}(1 - a_p(f)p^{-s} + \omega_f(p)p^{k-1-2s})^{-1} \prod_{p\mid N}(1 - a_p(f)p^{-s})^{-1}. \end{aligned}$$

It is important to note from the above examples that any well-behaved L-function is expected to satisfy the following fundamental properties (which might be challenging to prove):

(1) A meromorphic continuation to the entire complex plane;

(2) A functional equation relating s and $k - s$ for some $k \in \mathbb{R}$;

(3) An Euler product.

2. Special values of L-functions

There exists significant interest in the special values of L-functions, with deep results and conjectures linking them to crucial arithmetic information.

2.1. Class number formula. A prototypical example of such a relationship is the class number formula:

THEOREM 2.1. *Let F be a number field with r_1 real embeddings, r_2 pairs of complex embeddings, w roots of unity, discriminant D, and regulator R. The zeta function ζ_F has a simple pole at $s = 1$ with residue*

$$\operatorname{res}_{s=1} \zeta_F(s) = \frac{2^{r_1}(2\pi)^{r_2}R}{w\sqrt{|D|}} h_F,$$

where h_F is the class number of F.

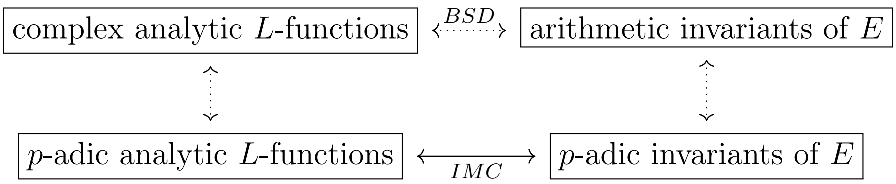
On the left-hand side, we encounter a special value of a complex meromorphic function from the domain of analysis. On the right-hand side, we find invariants associated with a number field which are more of arithmetic importance. Thus, the class number formula establishes an elegant connection between two distinct types of objects arising from number theory.

2.2. Birch and Swinnerton-Dyer (BSD) conjecture. A second famous example is the *Birch and Swinnerton-Dyer (BSD)* conjecture, which predicts a link between special values of L-functions of elliptic curves and arithmetic information. Let $E/\mathbb{Q}$ be an elliptic curve. The set of rational points $E(\mathbb{Q})$ forms a finitely generated abelian group, and Birch and Swinnerton-Dyer predicted that

$$\operatorname{ord}_{s=1} L(E,s) = \operatorname{rank}_{\mathbb{Z}} E(\mathbb{Q}). \tag{1}$$

Again, this prediction correlates two inherently different branches of mathematics: the left hand side is of analytic nature and the right hand side is of arithmetic nature. We should also mention that studying the properties of the L-function using the arithmetic properties of the elliptic curves is still considerably difficult. In fact, the only proof which proves the analytic continuation of the left hand side L-function uses Wile's modularity theorem.

2.3. Iwasawa main conjectures. The complete *BSD* conjecture remains an open problem. In Iwasawa theory, one of the main objectives is to explore and establish p-adic analogues of the *BSD*. This approach replaces the use of complex analysis, which is often ill-suited for arithmetic considerations, with p-adic analysis, where arithmetic naturally arises. For each prime number p, there exists a p-adic Iwasawa Main Conjecture (IMC) for an elliptic curve E. This conjecture establishes a connection between a p-adic analytic L-function and specific p-adic arithmetic invariants of E.



The tools available to address the p-adic side of this conjectural diagram far outnumber those available for the complex side. These tools include Euler systems, p-adic families and eigenvarieties, p-adic Hodge theory, and (φ, Γ)-modules. As a result, the p-adic conjectures are considerably more approachable than their complex counterparts. While the classical BSD conjecture in the complex setting remains open, its p-adic analogue, as captured by the *IMC* for elliptic curves, has been proven in numerous instances by Skinner and Urban (see [**SU13**]), building upon the groundwork laid by Kato (see [**Kat04**]).

3. p-adic Interpolation of Special Values of L-functions

As we have seen in the previous section, in modern number theory p-adic L-functions (p-adic analogues of complex L-functions) play a central role, as they tend to reflect arithmetic properties more directly than their complex counterparts. Since the discovery of the p-adic Riemann ζ-function by Kubota and Leopoldt in the 1960's, their deep arithmetic significance became apparent in Iwasawa's work on cyclotomic fields, which culminated in Mazur–Wiles' proof of the cyclotomic Iwasawa main conjecture. The primary way in which p-adic L-functions are defined is via the interpolation of the special values of classical L-values (i.e. values of usual L-functions).

3.1. Motivation. The motivation behind p-adic interpolation is to extend the concept of L-functions from the complex analytic setting to the p-adic setting. This is driven by the goal of understanding arithmetic properties of special values of L-functions using p-adic methods. Roughly speaking, a p-adic L-function is a p-adic analytic function that interpolates the values of a classical complex L-function at certain critical points. These p-adic L-functions capture arithmetic information that can be studied using p-adic techniques, which are often more accessible or offer different insights compared to their complex counterparts.

3.2. Example: Kubota-Leopoldt L-function. The complex $\zeta : \mathbb{C} \to \mathbb{C}$ function with complex analytic properties is rational at negative integers. Since $\mathbb{Z}$ is common to both $\mathbb{C}$ and $\mathbb{Z}_p \subset \mathbb{C}_p$, a natural question arises whether there exists a function

$$\zeta_p : \mathbb{Z}_p \to \mathbb{C}_p,$$

that is p-adic analytic and which agrees with the complex L-function at negative integers in the sense that

$$\zeta_p(1-n) = \Omega \cdot \zeta(1-n)$$

for some explicit factor Ω. We would say that such a function 'p-adically interpolates the special values of $\zeta(s)$', and these special values uniquely characterise ζ_p.

THEOREM 3.1 (Kubota-Leopoldt). *There exists a unique p-adic meromorphic function $\zeta_p : \mathbb{Z}_p \to \mathbb{C}_p$, such that for $k \in \mathbb{Z}_{>0}$ we have the following interpolation property:*

$$\zeta_p(1-k) = (1-p^{k-1})\zeta(1-k).$$

3.3. Conclusion. The p-adic interpolation of special values of L-functions offers a powerful tool for understanding the deep arithmetic properties of these functions. By transitioning from the complex to the p-adic setting, one can leverage p-adic techniques to gain new insights and solve longstanding problems in number theory.

4. Rankin-Selberg Convolution

Let N be an arbitrary positive integer. Consider f to be a cusp form of weight $k \geq 2$ for the level $\Gamma_0(N)$ with Dirichlet character ψ modulo N. Additionally, suppose that f is a primitive form. Let g be a modular form of weight $l < k$ for $\Gamma_0(N)$, with character ω. Suppose the q-expansions of f and g are given by:

$$f = \sum_{n=1}^{\infty} a(n)q^n, \quad g = \sum_{n=0}^{\infty} b(n)q^n.$$

Their associated L-functions are:

$$L(f, s) = \sum_{n=1}^{\infty} a(n)n^{-s}, \quad L(g, s) = \sum_{n=1}^{\infty} b(n)n^{-s}.$$

We define the Rankin-Selberg convolution L-function of f and g by:

$$L_N(f \otimes g, s) = L_N(2s + 2 - k - l, \omega\psi) \sum_{n=1}^{\infty} a(n)b(n)n^{-s},$$

where $L_N(2s+2-k-l, \omega\psi)$ denotes the Dirichlet L-series of $\omega\psi$ with Euler factors at the primes dividing N removed from its Euler product. This is a nicely behaved L function, in the sense that $L_N(f \otimes g, s)$ has an analytic continuation over the entire complex plane as a function of s, it admits an Eulerian product, and has a functional equation (for more details on this see [**Sad12**]).

The L-series obtained via Rankin-Selberg convolution is an important object to study, as we briefly describe now. It is well known that a Galois representation ρ can be associated with an L-function $L(\rho, s)$. A Galois representation ρ is said to be modular if $L(\rho, s)$ is "equal" to $L(f, s)$ for some modular form f. The 2-dimensional representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ that are geometric are conjectured to arise from modular forms in this manner. Specifically, it is expected that if ρ is an odd 2-dimensional compatible system of ℓ-adic representations, then there exists a modular form f and an integer j such that:

$$L(\rho, s) = L(f, s + j).$$

We can construct higher-dimensional representations from those arising from modular forms. For example, given two representations ρ_1 and ρ_2, we have:

$$L(\rho_1 \oplus \rho_2, s) = L(\rho_1, s) \cdot L(\rho_2, s).$$

This L-function inherits its analytic properties from $L(\rho_1, s)$ and $L(\rho_2, s)$, which might not be particularly interesting. However, one can attempt to construct an L-series corresponding to the representation $\rho_1 \otimes \rho_2$. Suppose that ρ_1 and ρ_2 correspond to modular forms f and g, respectively. Then, it turns out that the L-series associated with $\rho_1 \otimes \rho_2$ is closely related to the series $\sum_{n=1}^{\infty} a(n)b(n)n^{-s}$, whose remarkable properties were first studied by Rankin and Selberg. In fact, we have the following exact equality:

$$L(\rho_1 \otimes \rho_2, s) = L(f \otimes g, s).$$

A fundamental question, first posed by Langlands, then arises: Is this Rankin-Selberg convolution modular? In other words, is the tensor product of two modular representations itself a modular representation? (see [**Ram00**] for more details). This question motivates further study of the properties of Rankin-Selberg convolution, particularly its special values.

4.1. Special values of Rankin-Selberg Convolution. Shimura proved the following fundamental result about the special values of Rankin-Selberg convolution.

THEOREM 4.1 ([**Shi76**], Th. 3). *Let* $f = \sum_{n\geq 1} a(n)q^n \in \mathcal{S}_k(\Gamma_0(N), \psi)$ *be a primitive cusp form, and* $g = \sum_{n\geq 1} b(n)q^n \in \mathcal{M}_l(\Gamma_0(N), \omega)$ *be a modular form of weight* $l < k$. *Let* K_f *(respectively* K_g*) be the smallest number field containing the q-expansion coefficients of* f *(respectively* g*). Then*

$$\frac{L_N(f \otimes g, s)}{\pi^{2m+1-l}\langle f, f\rangle_N} \in K_f K_g \text{ for all integers } m \text{ with } l \leq m < k,$$

where

$$\langle f, f\rangle_N = \int_\Phi \overline{f(z)} f(z) y^{k-2}\, dx\, dy,$$

with Φ *being a fundamental domain for the upper half-plane modulo* $\Gamma_1(N)$. *In particular, when the q-expansion coefficients* $b(n)$ *of* g *are algebraic numbers (note that the q-expansion coefficients of* f *are automatically algebraic because* f *is primitive), then*

$$\frac{L_N(f \otimes g, s)}{\pi^{2m+1-l}\langle f, f\rangle_N}$$

is algebraic for all integers m *with* $l \leq m < k$.

HEURISTICS OF THE PROOF. The main idea behind the proof of this result is a certain rationality relation between special values of the convolution function and the Petersson norm of f whenever f is a normalised Hecke eigenform, proven by Shimura [**Shi76**]. Firstly, we find an integral formula for the convolution using an unfolding argument introduced by Rankin in [**Ran39**]. The formula expresses the convolution as the integral of $\overline{f}g$ times a non-holomorphic Eisenstein series. Each special value of the convolution is thus expressed as the Petersson product with such a form. Finally, Shimura shows that the interaction of the Petersson product with these forms is "nice": we can find an orthogonal basis containing f of a space in which that form lies, allowing us to relate the product to the norm of f using simple linear algebra.

5. p-adic interpolation of Rankin-Selberg convolution

As mentioned at the beginning of this chapter, our goal is to study the p-adic nature of the special values of the Rankin-Selberg convolution of two modular forms, scaled by a certain explicit factor:

$$\frac{L_N(f \otimes g, s)}{\pi^{2m+1-l}\langle f, f\rangle_N} \quad \text{for all integers } m \text{ with } l \leq m < k.$$

One important consequence of the algebraicity of these special values is that we can consider them as elements in some finite extension of $\mathbb{Q}_p$ after fixing (once for all) an embedding $\iota : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$.

In this section, we present Hida's p-adic interpolation of these special values of the Rankin-Selberg convolution. First, we introduce some additional terminology and notation needed to state Hida's theorem [**Hid85**].

Recall that $f = \sum_{n \geq 1} a(n)q^n$ is a primitive cusp form of weight $k \geq 2$. Let C be the conductor of f and suppose that f is with character ψ modulo C. We fix this f throughout this thesis. Since f is primitive, its q-expansion coefficients are algebraic, so $\iota(a(n)) \in \mathbb{C}_p$. We also assume that $\iota(a(p)) \in \mathbb{C}_p$ is a unit in $\mathbb{C}_p$. Consider the polynomial

$$X^2 - \iota(a(p))X + \iota(\psi(p))p^{k-1},$$

which has a unique root in $\mathbb{C}_p$ that is not a unit. If $p \mid C$, then $\gamma = 0$. Let $f_0(z)$ be the *stabilized* modular form obtained from f:

$$f_0(z) := \begin{cases} f(z) & \text{if } p \mid C, \\ f(z) - \iota^{-1}(\gamma) f(pz) & \text{if } p \nmid C. \end{cases}$$

It is well known that $f_0(z)$ is a common eigenform of all Hecke operators $T(n)$ for $n \geq 1$ of level pC, including those with n dividing pC. We say that a form is p-ordinary if p divides the level and its p-th q-expansion coefficient is a unit in $\mathbb{C}_p$. Then, $f_0(z)$ is a unique ordinary form of level pC with the same n-th Fourier coefficient as $f(z)$ for every n prime to p. Let C_0 be the smallest possible level of f_0:

$$C_0 = \begin{cases} C & \text{if } p \mid C, \\ pC & \text{if } p \nmid C. \end{cases}$$

Let $g = \sum_{n=0}^{\infty} b(n)q^n$ be an arbitrary modular form of weight $l < k$ of level $\Gamma_0(N)$ with character ω, and assume that $b(n) \in \overline{\mathbb{Q}}$ for all $n \geq 0$.

Further, we define non-negative integers μ, λ, C', and N' by

$$C_0 = C'p^\mu, \quad N = N'p^\lambda, \quad (C',p) = (N',p) = 1.$$

Put

$$Y = \varprojlim_v \mathbb{Z}/Np^v\mathbb{Z} = (\mathbb{Z}/N'\mathbb{Z}) \times \mathbb{Z}_p,$$
$$Y^\times = \varprojlim_v (\mathbb{Z}/Np^v\mathbb{Z})^\times = (\mathbb{Z}/N'\mathbb{Z})^\times \times \mathbb{Z}_p^\times.$$

It follows from Tychonoff's theorem that Y is a compact ring, and $Y^\times$ is a compact group. Let $\phi : Y \to \overline{\mathbb{Q}}$ be an arbitrary locally constant function on Y with the property that there is a character χ of finite order of the group $Y^\times$ such that

$$\phi(zy) = \chi(z)\phi(y) \quad \text{for all } z \in Y^\times \text{ and } y \in Y.$$

We then define the twist of g by ϕ as

$$g(\phi) = \sum_{n=0}^\infty \phi(n)b(n)q^n.$$

Put $\xi = \chi^2\omega$. It is known that $g(\phi)$ belongs to $\mathcal{M}_k(\Gamma_0(NN'p^\beta), r)$ for a sufficiently large $\beta > 1$. Now we fix such a $\beta > 1$. We assume that

$$C' \mid N',$$

and write

$$\gamma = \begin{pmatrix} N'^2/C' & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \tau_\beta = \begin{pmatrix} 0 & -1 \\ NN'p^\beta & 0 \end{pmatrix}.$$

THEOREM 5.1 ([**Hid85**], Theorem 2.2). *For each integer $b > 1$ prime to Np, there exists a unique bounded measure φ_b on Y with values in Ω satisfying the following property: for each non-negative integer r with $0 \le r < (k-l)/2$, let $j = l + 2r$. The value of the p-adic integral*

$$\int_Y \phi(y) y_p^r \, d\varphi_b(y)$$

is given by the image under $\iota : \overline{\mathbb{Q}} \to \mathbb{C}_p$ of

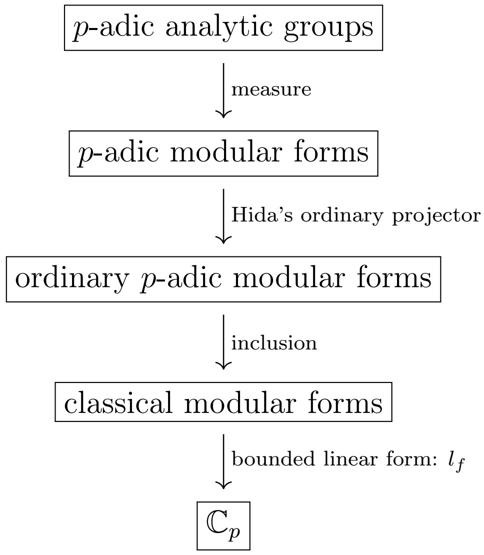
$$t(1 - b^{k-j}\psi\overline{\xi}(b))a(p,f_0)^{\mu-\lambda} \frac{L_{NN'p^\beta}(f_0[\gamma]_k \otimes g(\phi)[\tau_\beta]_l, j-r)}{\pi^{j+1}\langle h, f_0\rangle_{C_0}},$$

where y_p is the p-adic part of $y \in (\mathbb{Z}/N'\mathbb{Z}) \times \mathbb{Z}_p$,

$$h = f_0^\rho \left[\begin{pmatrix} 0 & -1 \\ C_0 & 0 \end{pmatrix} \right]_k,$$

and

$$t = t(r,\beta) = 2^{1-k-j} i^{k+j} p^{(\mu-\lambda)(1-k/2)+\beta j/2} (NN')^{(j-k)/2+1} \Gamma(j-r)\Gamma(r+1).$$


p-adic analytic groups
measure
p-adic modular forms
Hida's ordinary projector
ordinary p-adic modular forms
inclusion
classical modular forms
bounded linear form: l_f
$\mathbb{C}_p$

CHAPTER 2

p-adic modular forms and Hecke algebra

The study of p-adic modular forms was initiated by Serre, Katz, and Dwork in the early 1970s. The initial motivation for this theory stemmed from the problem of p-adic interpolation of the special values of the Riemann zeta function. In his 1973 paper, Serre defined p-adic modular forms as p-adic limits of q-expansions of classical modular forms of varying weights. He constructed p-adic L-functions using these families of p-adic modular forms.

Katz provided a modular definition of Serre's p-adic modular forms of integral weight. These forms are defined as specific functions on the moduli space of test objects, which consist of ordinary elliptic curves with a level structure. Katz also offered modular descriptions of the action of Hecke operators on these modular forms, including the analogue of Atkin's classical U_p operator, referred to as the U operator. This operator transforms a modular form with q-expansion $\sum_n a(n)q^n$ into $\sum_n a(np)q^n$.

Extensive amount of literature is available on this subject, the original papers of Serre and Katz still stand as a good reference. However, in this text we refrain from giving extensive details and stick to the definitions given by Hida's which are equivalent to those of Serre and Katz.

1. p-adic modular forms

Let p be the same prime from chapter 1. Let $\mathbb{Q}_p$ be the field of p-adic numbers with the normaized p-adic absolute value $|p|_p = 1/p$, and let $\overline{\mathbb{Q}}_p$ be an algebraic closure of $\mathbb{Q}_p$. We know that $\overline{\mathbb{Q}}_p$ is not complete. We denote by $\mathbb{C}_p$ the p-adic completion of an algebraic closure of $\mathbb{Q}_p$. Under the fixed embedding: $\iota : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$, the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ in $\mathbb{C}$ can also be regarded as a subfield of $\mathbb{C}_p$. Any extension of $\mathbb{Q}_p$ will be considered in $\mathbb{C}_p$. As a result, there is a p-adic valuation v_p on $\overline{\mathbb{Q}}$ given by the restriction of that of $\mathbb{C}_p$.

In this section we present the definition of p-adic modular forms according to Hida ([**Hid85**]). For each power series $g = \sum_{n=0}^{\infty} a_n q^n \in \mathbb{C}_p[[q]]$

with coefficients in $\mathbb{C}_p$, we can define its p-adic norm by

$$|g|_p := \sup_n |a_n|_p.$$

Thus, given a modular form $f(z) = \sum_{n\geq 0} a(n,f)q^n$ ($q = e^{2\pi iz}$) with *algebraic* Fourier coefficients, we can speak of its norm through its q-expansion: let $f = \sum_{n=0}^{\infty} a(n,f)q^n$ be its q-expansion, then define its p-adic norm by

$$|f|_p := \sup_n \left|a(n,f)\right|_p.$$

Then the norm $|f|_p$ is known to be a well-defined real number because of the following result of Shimura.

THEOREM 1.1 (**[Shi02]**). *Let $\mathbb{A}$ be the ring of all algebraic integers in $\overline{\mathbb{Q}}$, and Γ be any congruence subgroup. Then, for every $f \in \mathcal{M}_k(\Gamma, \overline{\mathbb{Q}})$, there exists $\alpha \in \mathbb{Z}_{>0}$ such that $\alpha f \in \mathcal{M}_k(\Gamma, \mathbb{A})$.*

Indeed, if f is a modular form with algebraic coefficients, then there is a positive rational α such that αf has integral coefficients, thus the norm $|\alpha f|_p \leq 1$, which implies that $|f|_p \leq \frac{1}{|\alpha|_p}$.

1.1. Definition. Let $N \in \mathbb{Z}_{>0}$. For a subring A of $\overline{\mathbb{Q}}$, we denote by $\mathcal{M}_k(\Gamma_1(N), A)$ the subset of $\mathcal{M}_k(\Gamma_1(N))$ comprising modular forms with Fourier coefficients belonging to A, then $\mathcal{M}_k(\Gamma_1(N), A)$ is an A-module. Similarly, for any Dirichlet character ψ modulo N, we define $\mathcal{M}_k(\Gamma_0(N), \psi, A)$ as the subset of A-rational modular forms within $\mathcal{M}_k(\Gamma_0(N), \psi)$, constituting an A-module as well. Analogously, we extend this notion to define the A-modules of cusp forms, denoted by $\mathcal{S}_k(\Gamma_1(N), A)$ and $\mathcal{S}_k(\Gamma_0(N), \psi, A)$.

Let K_0 be a number field and K be its topological closure in $\mathbb{C}_p$. Then the spaces $\mathcal{M}_k(\Gamma_1(N), K_0)$ and $\mathcal{M}_k(\Gamma_0(N), \chi; K_0)$ can be thought of living inside $\mathbb{C}_p[[q]]$. Let $\psi : \mathbb{Z} \to K_0$ be a Dirichlet character modulo N with values in K_0. Put,

$$\begin{aligned}
\mathcal{M}_k(\Gamma_1(N); K) &= \mathcal{M}_k(\Gamma_1(N); K_0) \otimes_{K_0} K, \\
\mathcal{S}_k(\Gamma_1(N); K) &= \mathcal{S}_k(\Gamma_1(N); K_0) \otimes_{K_0} K, \\
\mathcal{M}_k(\Gamma_0(N), \psi; K) &= \mathcal{M}_k(\Gamma_0(N), \psi; K_0) \otimes_{K_0} K, \\
\mathcal{S}_k(\Gamma_0(N), \psi; K) &= \mathcal{S}_k(\Gamma_0(N), \psi; K_0) \otimes_{K_0} K.
\end{aligned}$$

Then these spaces are finite dimensional Banach spaces over K and are *independent* of the choice of the dense subfield K_0 of K. By considering the q-expansion, the space $\mathcal{M}_k(\Gamma_1(N), K_0)$ is naturally embedded in the power series ring $K_0[[q]]$, and hence we may consider $\mathcal{M}_k(\Gamma_1(N), K)$ as the K-linear span of $\mathcal{M}_k(\Gamma_1(N), K_0)$ in $K[[q]]$. Thus every element of $\mathcal{M}_k(\Gamma_1(N), K)$ has a unique q-expansion, which will be written as $f =$

$\sum_{n\geq 0} a(n,f)q^n \in K[[q]]$, the norm of f is again given by $\sup_n \left|a(n,f)\right|_p$. Let $\mathcal{O}_K$ be the valuation ring of K, and define

$$\begin{aligned}\mathcal{M}_k(\Gamma_1(N);\mathcal{O}_K) &= \{f \in \mathcal{M}_k(\Gamma_1(N);K) : |f|_p \leq 1\} \\ &= \mathcal{M}_k(\Gamma_1(N);K) \cap \mathcal{O}_K[[q]], \\ \mathcal{M}_k(\Gamma_0(N),\psi;\mathcal{O}_K) &= \{f \in \mathcal{M}_k(\Gamma_0(N),\psi;K) : |f|_p \leq 1\}.\end{aligned}$$

These spaces are complete normed $\mathcal{O}_K$-modules of finite rank.

Let $N \in \mathbb{Z}_{>0}$ be arbitrary, and ψ be a character modulo N. Let A denote either of K or $\mathcal{O}_K$. Let $r > s \geq 1$ be two arbitrary integers. Then we have

$$\mathcal{M}_k(\Gamma_1(Np^s);A) \subset \mathcal{M}_k(\Gamma_1(Np^r);A).$$

This forms a directed system, hence we can take direct limit. Define

$$\begin{aligned}\mathcal{M}_k(N;A) &= \varinjlim_r \mathcal{M}_k(\Gamma_1(Np^r);A), \\ \mathcal{M}_k(N,\psi;A) &= \varinjlim_r \mathcal{M}_k(\Gamma_0(Np^r),\psi;A).\end{aligned}$$

Clearly, these spaces do not depend on the p-primary part of N. Let $\overline{\mathcal{M}}(N;A)$ denote the completion of $\mathcal{M}_k(N;A)$ with respect to the norm $|\cdot|_p$. Similarly $\overline{\mathcal{M}}_k(N,\psi;A))$ be the completion of $\mathcal{M}_k(N,\psi;A)$.

Definition 1.1 (p-adic modular forms). A **p-adic modular form** is an element of $\overline{\mathcal{M}}(N;A)$.

Remark. The suffix "k" is omitted in the notation $\overline{\mathcal{M}}(N;A)$ because this space, as a subspace of $A[[q]]$, is determined independently of the weight k when $k > 2$. Although this fact is implicitly covered in the works of Katz and Serre on p-adic modular forms, we won't delve into the details here since it is not necessary for our purposes.

2. Hecke Algebras

We now extend the action of Hecke operators from classical modular forms to p-adic modular forms.

Let $Z = \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$, and write each element $z \in Z$ as a pair $z = (z_p, z_0)$ where $z_p \in \mathbb{Z}_p^\times$ and $z_0 \in (\mathbb{Z}/N\mathbb{Z})^\times$. The group Z is a topological group with the product topology, combining the usual topology on $\mathbb{Z}_p^\times$ and the discrete topology on $(\mathbb{Z}/N\mathbb{Z})^\times$.

We define the action of the topological group Z on $\mathcal{M}_k(\Gamma_1(Np^r);\mathbb{C}_p)$ as follows: for each $z \in Z$, choose an element $\sigma_z \in \mathrm{SL}_2(\mathbb{Z})$ such that

$$\sigma_z \equiv \begin{pmatrix} * & * \\ 0 & z \end{pmatrix} \pmod{Np^r}.$$

Then, for $f \in \mathcal{M}_k(\Gamma_1(Np^r); \mathbb{C}_p)$, define

$$f \mid z := z_p^k(f[\sigma_z]_k).$$

Let ℓ be any prime (which may or may not equal p). If ℓ is coprime to Np, then we can consider ℓ as an element of Z. For each prime ℓ, the Hecke operators $T(\ell)$ and $T(\ell,\ell)$ on $\mathcal{M}_k(\Gamma_1(Np^r); \mathbb{C}_p)$ are defined as follows:

Let $f = \sum_{n\geq 0} a(n,f)q^n \in \mathcal{M}_k(\Gamma_1(Np^r); \mathbb{C}_p)$.

$$a(n, T(\ell)f) = \begin{cases} a(n\ell, f) + \frac{1}{\ell} a\left(\frac{n}{\ell}, f \mid \ell\right) & \text{if } \ell \nmid Np, \\ a(n\ell, f) & \text{if } \ell \mid Np, \end{cases}$$

$$a(n, T(\ell,\ell)f) = \begin{cases} \frac{1}{\ell^2} a(n, f \mid \ell) & \text{if } \ell \nmid Np, \\ 0 & \text{if } \ell \mid Np. \end{cases}$$

Throughout the remaining section, we denote by A one of the rings K or its valuation ring $\mathcal{O}_K$. It is well known that the Hecke operators $T(\ell)$ and $T(\ell,\ell)$ preserve the space of A-rational modular forms (see **[Hid86]** for more details). Furthermore, it can be clearly checked that the action of $T(\ell)$ and $T(\ell,\ell)$ is uniformly continuous.

Definition 2.1. Let $\mathcal{H}_k(\Gamma_1(Np^r); A)$ be the A-subalgebra of the ring of all A-linear endomorphisms of $\mathcal{M}_k(\Gamma_1(Np^r); A)$ generated by $T(\ell)$ and $T(\ell,\ell)$ for all primes ℓ, called the Hecke algebra for the space $\mathcal{M}_k(\Gamma_1(Np^r); A)$. Similarly, we define the Hecke algebra $\mathfrak{H}_k(\Gamma_1(Np^r); A)$ for the space of cusp forms $\mathcal{S}_k(\Gamma_1(Np^r); A)$.

Let $r > s \geq 1$ be two integers. Then we have the following commutative diagrams:

$$\begin{array}{ccc} \mathcal{M}_k(\Gamma_1(Np^s); A) & \hookrightarrow & \mathcal{M}_k(\Gamma_1(Np^r); A) \\ \big\downarrow {\scriptstyle T(\ell) \text{ resp. } T(\ell,\ell)} & & \big\downarrow {\scriptstyle T(\ell) \text{ resp. } T(\ell,\ell)} \\ \mathcal{M}_k(\Gamma_1(Np^s); A) & \hookrightarrow & \mathcal{M}_k(\Gamma_1(Np^r); A) \end{array}$$

Here, the horizontal arrows represent natural inclusions. Therefore, the restriction of operators in $\mathcal{H}_k(\Gamma_1(Np^r); A)$ to $\mathcal{H}_k(\Gamma_1(Np^s); A)$ provides a surjective A-algebra homomorphism from $\mathcal{H}_k(\Gamma_1(Np^r); A)$ onto $\mathcal{H}_k(\Gamma_1(Np^s); A)$. Consequently, the following inverse limits are well-defined:

$$\begin{aligned} \mathcal{H}(N;A) &= \varprojlim_r \mathcal{H}_k(\Gamma_1(Np^r);A), \\ \mathcal{H}_k(N,\psi;A) &= \varprojlim_r \mathcal{H}_k(\Gamma_0(Np^r,\psi);A), \end{aligned} \tag{2}$$

which naturally act on $\mathcal{M}_k(N;A)$ and $\mathcal{M}_k(N,\psi;A)$ for $A = K$ or $\mathcal{O}_K$.

Since the action of $T(\ell)$ and $T(\ell,\ell)$ on $\mathcal{M}_k(\Gamma_1(Np^r);\mathbb{C}_p)$ is uniformly continuous for each r, it extends to a uniformly continuous action on $\mathcal{M}_k(N;A)$. Furthermore, because elements of $\mathcal{O}_K$ have norm ≤ 1, any $\mathcal{O}_K$-linear combination of $T(\ell)$ and $T(\ell,\ell)$ also acts uniformly continuously on $\mathcal{M}_k(N;A)$ for $A = K$ or $\mathcal{O}_K$. Thus, the action of $\mathcal{H}_k(N;\mathcal{O}_K)$ (resp. $\mathcal{H}_k(N,\psi;\mathcal{O}_K)$) can be naturally extended to an action on $\overline{\mathcal{M}}(N;A)$ (resp. $\overline{\mathcal{M}}_k(N,\psi;A)$).

Definition 2.2. The $\mathcal{O}_K$-algebra $\mathcal{H}_k(N;\mathcal{O}_K)$ (resp. $\mathcal{H}_k(N,\psi;\mathcal{O}_K)$) is called the **Hecke algebra** of $\overline{\mathcal{M}}(N;A)$ (resp. $\overline{\mathcal{M}}_k(N,\psi;A)$).

3. Space of ordinary forms

One of the cornerstone theorems in complex analysis states that every complex holomorphic function is also complex analytic. However, this equivalence does not hold for real holomorphic functions; in the realm of real analysis, holomorphic functions form a subset of analytic functions. Given the superior behavior of holomorphic functions compared to their analytic counterparts, it is logical to focus on the properties specific to holomorphic functions.

Recognizing a similar pattern in the transition from an analytic to an algebraic setting, Hida observed that within the domain of p-adic modular forms, certain forms known as ordinary forms exhibit more favorable behavior than standard modular forms. In this section, we define the subspace of ordinary forms and describe some of its properties, which will be prominently featured in the subsequent sections.

3.1. The ordinary projector.

Lemma 3.1. *Let K and $\mathcal{O}_K$ be as above. For any commutative $\mathcal{O}_K$-algebra A of finite rank over $\mathcal{O}_K$ and for any $x \in A$, the limit*

$$\lim_{n\to\infty} x^{n!}$$

exists in A and is an idempotent of A.

Proof. See [[**Hid93**], Lem. 7.2.1]. ∎

In the Hecke algebra $\mathcal{H}_k(\Gamma_0(Np^r), \psi; \mathcal{O}_K)$ (resp. $\mathfrak{H}_k(\Gamma_0(Np^r), \psi; \mathcal{O}_K)$), we have the operator $U(p) := T(p)$, and thanks to the above lemma we can define

$$e_r := \lim_{n\to\infty} U(p)^{n!}.$$

The operator e_r defined above is called *Hida ordinary projector* for $\mathcal{M}_k(\Gamma_0(Np^r), \psi; \mathcal{O}_K)$.

DEFINITION 3.1. We say that a modular form $f \in \mathcal{M}_k(\Gamma_0(Np^r), \psi; \mathcal{O}_K)$ is p-ordinary if: $e_r \cdot f = f$.

We should mention here that a necessary condition for a form to be p-ordinary is that p should divide its level, but here $r \geq 1$, so the condition is satisfied automatically.

REMARK. If $f \in \mathcal{M}_k(\Gamma_0(Np^r), \psi; \mathcal{O}_K)$ is an eigenform of $U(p)$ with eigenvalue $\lambda \in \overline{\mathbb{Q}_p}$, it is easy to see that

$$e_r \cdot f = \begin{cases} f & \text{if } |\lambda| = 1, \\ 0 & \text{if } |\lambda| = 0. \end{cases}$$

Thus the above definition is equivalent to the definition that we mentioned in section 5. As for a primitive cusp form $f = \sum_{n\geq 1} a(n)q^n$,

$$a(p) \text{ is a unit in } \mathbb{C}_p \Leftrightarrow e.f = f,$$

where e is the idempotent as constructed above in a suitable Hecke algebra.

It is clear that the idempotent e_r is plainly compatible with the projective limit 2. Thus we can define an idempotent e of $\mathcal{H}(N; A)$ by the projective limit

$$e := \varprojlim_r e_r.$$

DEFINITION 3.2 (Ordinary part). For any module $\mathcal{M}$ over these Hecke algebras, we define the ordinary part $\mathcal{M}^o$ of $\mathcal{M}$ to be the corresponding component $e\mathcal{M}$ for the idempotent e.

For any $r \geq s \geq 0$, we put

$$\Phi_r^s := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{Np^r}, a \equiv d \equiv 1 \pmod{Np^s} \right\}.$$

PROPOSITION 3.1. (1) *Suppose $s > 0$ and $r - s \geq m$. Then,*

$$\Phi_r^s \begin{pmatrix} 1 & 0 \\ 0p^m & \end{pmatrix} \Phi_s^r = \bigcup_{0\leq u<p^m} \Phi_r^s \begin{pmatrix} 1 & u \\ 0 & p^m \end{pmatrix}.$$

(2) *Again, suppose* $s > 0$ *and* $r - s \geq m$. *Then,*

$$\Phi_r^s \begin{pmatrix} 1 & 0 \\ 0 & p^m \end{pmatrix} \Phi_s^r = \Phi_r^s \begin{pmatrix} 1 & 0 \\ 0 & p^m \end{pmatrix} \Phi_{r-m}^s.$$

(3) *Let* A *be one of* K *or* $\mathcal{O}_K$ *and* $r \geq 1$ *be a positive integer, and* ψ *be a (not necessarily primitive) Dirichlet character modulo* Np^{r-1}. *If* $f(z) \in \mathcal{M}_k(\Gamma_0(Np^r), \psi; R)$, *then* $U(p)f \in \mathcal{M}_k(\Gamma_0(Np^{n-1}), \psi; R)$. *Similar statement holds for cusp forms.*

Let $C(\psi)$ be the conductor of the Dirichelt character ψ and define positve integers N' and s by:

$$N = N'p^r,\ C(\psi) = C'(\psi)p^{s'} \text{ and } (N', p) = (C'(\psi), p) = 1,$$

put $s = \max(s', 1)$.

Theorem 3.2 (Ordinary forms as classical forms). *The ordinary part* $\overline{M}_k^o(N, \psi; \mathcal{O}_K)$ *of* $\overline{M}_k(N, \psi; \mathcal{O}_K)$ *is a finite rank* $\mathcal{O}_K$ *module. Moreover,* $\overline{M}_k^o(N, \psi; \mathcal{O}_K)$ *is contained in* $\mathcal{M}_k(\Gamma_0(N'p^s), \psi; \mathcal{O}_K)$.

Proof. For simplicity assume that $p \nmid N$. For each $r \geq 0$, there is a sufficiently large m_r such that

$$U(p)^{m_r} \mathcal{M}_k(\Gamma_0(Np^r), \psi; \mathcal{O}_K) \subset \mathcal{M}_k(\Gamma_0(Np), \psi; \mathcal{O}_K).$$

Thus,

$$e_r \cdot \mathcal{M}_k(\Gamma_0(Np^r), \psi; \mathcal{O}_K) \subset \mathcal{M}_k(\Gamma_0(Np), \psi; \mathcal{O}_K).$$

Clearly,

$$\operatorname{rank}_{\mathcal{O}_K} \overline{\mathcal{M}}_k^o(N, \psi; \mathcal{O}_K) \leq \operatorname{rank}_{\mathcal{O}_K} \mathcal{M}_k(Np, \psi; \mathcal{O}_K).$$

Since, $\mathcal{M}_k(Np, \psi; \mathcal{O}_K)$ is free of finite rank, the result follows. ■

4. A construction of a linear form

In all seminal works (Katz, Hida), a function from modular forms to p-adic numbers is constructed.

- Katz's approach, being the first, is perhaps the simplest: he extracts the constant term of the modular form.
- Hida's methodology resembles his treatment of Hida families, where he employs a similar concept. Specifically, he utilizes his operator: the ordinary projector e, which represents the limit of $T_p^{p^n}$ or a related construction, and crucially, this limit exists for ordinary forms. When e is applied to an ordinary form g (of level Np^∞), the result is expected to be a multiple of a fixed ordinary form f, i.e., $eg = a_g f$. Though details may vary, the essence remains consistent.

- We construct a linear map from the set of modular forms to a number field using certain properties of the Hecke operators.

Let $f = \sum_{n\geq 1} a(n,f)q^n$ be a primitive form of conductor C, of weight $k \geq 2$ and character ψ modulo C. Also assume that, $\left|a(p,f)\right|_p = 1$. Now we are ready to define a continuous linear form l_f (attached to f) on $\overline{\mathcal{M}}_k(C,\psi;K)$ into K.

LEMMA 4.1. *Let f be as above. Then, there is a unique ordinary form f_0 of weight k such that $a(n,f_0) = a(n,f)$ for all n coprime to p. Moreover, f_0 is explicitly given by*

$$f_0(z) = \begin{cases} f(z) & p \mid C, \\ f(z) - \alpha_0 f(pz) & p \nmid C, \end{cases}$$

where α_0 is the unique root of $X^2 - a(p,f)X + \psi(p)p^{k-1}$ with $|\alpha_0|_p < 1$. Furthermore, if α_1 is another root of this polynomial, put $f_1(z) = f(z) - \alpha_1 f(pz)$, then

$$U(p)f_i = \alpha_i f_i, \quad i = 0,1.$$

PROOF. See Lemma 3.3 in [**Hid85**]. ■

Let f_0 be the unique p-ordinary form associated to f. Then it is well known that f_0 is a common eigenform of all Hecke operators $T(n)$ for $n \geq 1$ of level pC, inclusing those with n dividing pC. Let C_0 be the conductor of f_0 :

$$C_0 = \begin{cases} C & \text{if } p \mid C, \\ pC & \text{if } p \nmid C. \end{cases}$$

Since $C \mid C_0$, we have $\overline{\mathcal{M}}_k(C,\psi;K) \subset \overline{\mathcal{M}}_k(C_0,\psi;K)$. Thus we can focus on constructing a linear form l_f on $\overline{\mathcal{M}}_k(C_0,\psi;K)$ into K. It is also clear from theorem 3.2 that the ordinary projector e sends $\overline{\mathcal{M}}_k(C_0,\psi;K)$ into $\mathcal{M}_k(C_0,\psi;K)$. We construct a linear form from $\mathcal{M}_k(C_0,\psi;K)$ to K and compose it with e to finally get the desired linear form.

Idea. The idea is to construct a $\mathbb{C}$-basis of $\mathcal{M}_k(C_0,\psi)$ such that one of the basis elements is f_0. We get the desired linear form by sending $g \in \mathcal{M}_k(C_0,\psi)$ to the unique coefficient of f_0 in the decomposition of g in the above basis.

For any $g \in \mathcal{S}_k(\Gamma_0(C_0),\psi)$ we define $U(g) \subset \mathcal{S}_k(\Gamma_0(C_0,\psi)$ consisting of elements $G \in \mathcal{S}_k(\Gamma_0(C_0,\psi)$ such that

$$T(\ell)G = a(\ell,g)G \text{ for cofinitely many primes } \ell.$$

By the works of Hecke and Miyake, it is well known that there is an orthogonal decomposition

$$\mathcal{S}_k(\Gamma_0(C_0,\psi) = \bigoplus_{i=1}^{n} U(g_i),$$

for some $g_i \in \mathcal{S}_k(\Gamma_0(C_0,\psi)$, we may assume that g_i is an eigenform for almost all Hecke operators of level C_0, and $g_1 = f$.

PROPOSITION 4.1. *For any $0 \le n \in \mathbb{Z}$, put $f^{(n)}(z) = f(p^n z)$. Then $f^{(0)}, \ldots, f^{(v)}$ gives a $\mathbb{C}$-basis of $U(f)$.*

PROOF. See [**Miy71**]. ■

COROLLARY 4.1.1. *$U(f)$ is spanned by $f_0, f_1, f^{(2)}, \ldots, f^{(v)}$ over $\mathbb{C}$, where f_0 and f_1 are same as in lemma 4.1 .*

PROOF. Since f_0 and f_1 are linear spans of $f^{(0)}$ and $f^{(1)}$, the result follows. ■

PROPOSITION 4.2. *If K_0 is a large enough number field, put*

$$U_0 = K_0 \cdot f_0, \quad U_1 = K_0 \cdot f_1.$$

Then we have the following decomposition:

$$\mathcal{S}_k(\Gamma_0(C_0), \psi; K_0) = U_0 \oplus U_1 \oplus U_2,$$

where explicitly for some subspace U_2 of $\mathcal{S}_k(\Gamma_0(C_0,\psi;K_0))$.

This decomposition may not be sufficient as the Eisenstein measure can have values in $\mathcal{M}_k(\Gamma_0(C_0),\psi)$. We know that, if $\mathcal{N}_k(\Gamma_0(C_0),\psi)$ is the set of linear combinations of Eisenstein series, then we have the following orthogonal decomposition (see [**Miy06**]):

$$\mathcal{M}_k(\Gamma_0(C_0),\psi) = \mathcal{N}_k(\Gamma_0(C_0),\psi) \oplus \mathcal{S}_k(\Gamma_0(C_0),\psi).$$

Put $U_{-1} = \mathcal{N}_k(\Gamma_0(C_0),\psi) \cap \mathcal{M}_k(\Gamma_0(C_0),\psi;K_0)$. Then, we have the following decomposition:

$$\mathcal{M}_k(\Gamma_0(C_0),\psi;K_0) = U_{-1} \oplus U_0 \oplus U_1 \oplus U_2.$$

Let π_f be the following projection map:

$$\pi_f : \mathcal{M}_k(\Gamma_0(C_0),\psi;K_0) \to U_0.$$

Then we define the following linear form attached to f:

$$\lambda_f : \mathcal{M}_k(\Gamma_0(C_0),\psi;K_0) \to K_0, \qquad g \mapsto a(1,\pi_f(g)). \tag{3}$$

We can naturally extend this linear form λ_f to a linear form

$$\lambda_f : \mathcal{M}_k(\Gamma_0(C_0),\psi;K) \to K.$$

Then the linear form

$$l_f : \overline{\mathcal{M}}_k(C_0, \psi; K) \to K$$

is defined by

$$l_f(g) = a(1, \lambda_f(e \cdot g)), \quad g \in \overline{\mathcal{M}}_k(C_0, \psi; K).$$

4.1. Evaluation of l_f by Petersson inner product. Let $g_1, g_2 \in \mathcal{M}_k(\Gamma_0(C_0), \psi; K_0)$ such that

$$g_1 - g_2 \in \mathcal{N}_k(\Gamma_0(C_0), \psi; K_0),$$

then it is clear that $\lambda_f(g_1) = \lambda_f(g_2)$. Since Petersson inner product is non-degenerate on the cusp forms, there exists a unique $h_f \in \mathcal{S}_k(\Gamma_0(C_0), \psi; K_0)$ with the property that

$$\lambda_f(g) = \langle h_f, g \rangle_{C_0}, \quad \forall g \in \mathcal{M}_k(\Gamma_0(C_0), \psi; 0).$$

PROPOSITION 4.3. *Assume that K_0 contains all the Fourier coefficients of the ordinary form f_0. Then, the linear form l_f has values in the finite algebraic number field K_0 on $\mathcal{M}_k(\Gamma_0(C_0p^n), \psi; K)$ for evry $n \geq 0$. Furthermore, we have*

$$l_f(g) = a(p, f_0)^{-n} p^{n(k-1)} \frac{\langle h_{f,n}, g \rangle_{C_0 p^n}}{\langle h, f_0 \rangle_{C_0}} \quad g \in \mathcal{M}_k(\Gamma_0(C_0p^n), \psi; K_0),$$

where $h_{f,n}(z) = h_f(p^n z)$.

CHAPTER 3

Differential operators

In this chapter we introduce the theory of differential operators on modular forms, and state some related facts that we will need to define the convolution of p-adic Eisenstein measure and p-adic theta measure. Define the differential operators on H by

$$\delta_s = \frac{1}{2\pi i}\left(\frac{l}{2iy} + \frac{\partial}{\partial z}\right)$$

$$d = \frac{1}{2\pi i}\frac{\partial}{\partial z} = q\frac{\mathrm{d}}{\mathrm{d}q}, \quad q = \exp\{2\pi i z\}, z = x + iy,$$

$$\delta_s^{(r)} = \delta_{l+2r-2}\cdots\delta_{s+2}\delta_s \quad \text{for } 0 \le r \in \mathbb{Z},$$

where we put $\delta_s^{(0)} = 1$ as the identity operator. These operators satisfy the following relations

(4) $\quad \delta_{s+t}(fg) = g\delta_s(f) + f\delta_t(g)$ and $\delta_k^{(r)}(f[\gamma]_k) = (\delta_k^{(r)}f)[\gamma]_{k+2r}$

for $\gamma \in \mathrm{GL}_2^+(\mathbb{R})$ and every positive integer k. Furthermore, by induction on r we have the following:

(5) $$\delta_s^{(r)} = \sum_{0\le t\le r}\binom{r}{t}\frac{\Gamma(s+r)}{\Gamma(s+t)}(-4\pi y)^{t-r}d^t.$$

Lemma 0.1. *Let $f \in \mathcal{S}_k(\Gamma_0(N), \chi)$, and $g \in \mathcal{M}_k(N, \overline{\chi})$ and $k = l + 2r$ with a positive integer r. Then $\langle f_\rho, \delta_s^{(r)} g\rangle = 0$.*

Proof. We use induction on r: the base case is clear, since A_0 is just the space of modular forms. ∎

Definition 0.1. A **nearly holomorphic modular form** of weight k and level $\Gamma_1(N)$ is a continuous function h with the following properties:

- h transforms like a modular form of weight k and level $\Gamma_1(N)$.
- $h(z) = \sum_{v=0}^{r} y^{-v}g_v(z)$, where the g_v's are holomorphic functions on the upper half-plane with Fourier expansions

$$g_v(z) = \sum_{n=0}^{\infty} b_{vn}q^n.$$

Example 0.1. The Eisenstein series $E_2 = 1 - 24q - \frac{3}{\pi y} + O(q^2)$ of weight 2 is a nearly holomorphic modular form.

Let A_r be the set of all functions of the form $h(z) = \sum_{v=0}^{r} y^{-v} g_v(z)$, where the g_v's are holomorphic functions on the upper half-plane with Fourier expansions. That is, elements of A_r are polynomials of degree r in y^{-1}, with g_v's as coefficients. In particular, any nearly holomorphic modular form is in A_r for some r, which we call its degree of near holomorphy. The following theorem gives the structure of nearly holomorphic modular forms.

THEOREM 0.2. *Suppose that, with a positive integer $k > 2r$ and a Dirichlet character χ modular N, an element h of A_r satisfies*

- $h(\gamma(z))(cz+d)^{-k} \in A_r$ *for all* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$,
- $h(\gamma(z))(cz+d)^{-k} = \chi(d)h(z)$ *for all* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$.

Then $h(z) = \sum_{v=0}^{r} \delta_{k-2v}^{(v)} g_v$ *with elements* $g_v \in \mathcal{M}_{k-2v}(\Gamma_0(N), \chi)$, *which are uniquely determined by h. Furthermore, g_0 is called the* ***holomorphic projection*** *of h and denoted by $H(h)$.*

The statement roughly says that the nearly holomorphic modular forms can be expressed as linear combinations of usual modular forms to which we apply the δ_s operators. Thus, if h satisfies the two conditions of the above theorem, then

$$\langle f, h\rangle_N = \langle f, H(h)\rangle_N,$$

for every $f \in \mathcal{S}_k(\Gamma_0(N), \chi)$. This follows from lemma 0.1 and the property that $\langle f, h\rangle_N = -\langle f_\rho, h\rangle_N$. Here the Petersson inner product $\langle f, h\rangle_N$ is defined as usual, since h transforms under $\Gamma_0(N)$ as if it were an element of $\mathcal{M}_k(\Gamma_0(N), \chi)$.

Let K_0 be a subfield of $\overline{\mathbb{Q}}$ and l and m be positive integers. Let $g \in \mathcal{M}_l(\Gamma_0(N), \xi; K_0)$ and $h \in \mathcal{M}_m(\Gamma_0(N), \chi; K_0)$. Then it is easy to verify that $g\delta_m^{(r)}h$ satisfies the two conditions of theorem 0.2 thus we can write

$$g\delta_m^{(r)}h = H(g\delta_m^{(r)}h) + \sum_{v=1}^{r} \delta_{k-2v}^{(v)} g_v, \tag{6}$$

where $g_v \in \mathcal{M}_{k-2v}(\Gamma_0(N), \xi\chi; K_0)$ for $k = l + m + 2r$. We also have

$$\langle f, g\delta_m^{(r)}h\rangle_N = \langle f, H(g\delta_m^{(r)}h)\rangle_N,$$

for every element f of $\mathcal{S}_k(\Gamma_0(N), \xi\chi)$. Now we describe some useful properties of the holomorphic projection of $g\delta_m^{(r)}h$, which will be used to prove some properties of Hida's measure.

Lemma 0.3. *Let* $\mathcal{O}_{K_0} = \{x \in K_0 : |x|_p \leq 1\}$*, and suppose that* $g \in \mathcal{M}_l(N, \mathcal{O}_{K_0})$ *and* $h \in \mathcal{M}_m(N, \mathcal{O}_{K_0})$*. Then, we can find a positive inger* C *independently of* g *and* h *such that*

$$CH(g\delta_m^{(r)}h) \in \mathcal{M}_k(N, \mathcal{O}_{K_0}), \quad (k = l + m + 2r).$$

The integer C *depends only on* l, m *and* r*.*

Lemma 0.4. *Suppose that* $g \in \mathcal{M}_l(\Gamma_1(N); K_0)$ *and* $h \in \mathcal{M}_m(\Gamma_1(N); K_0)$*. For a positive integer* r *write*

$$g\delta_m^{(r)}h = \sum_{v=0}^{r} \delta_{k-2v}^{(v)} g_v,$$

where $g_v \in \mathcal{M}_{k-2v}(\Gamma_1(N); K_0)$*, and* $k = l + m + 2r$*. Put*

$$g' = -\sum_{n=0}^{r-1} d^{(n)} g_{n+1}.$$

Then the p*-adic norm* $\left|a(n, g')\right|_p$ *of the Fourier coefficients of* g' *is bounded for all* n*, and we have that*

$$H(g\delta_m^{(r)}h) = gd^{(r)}h + dg'.$$

Moreover, $H(g\delta_m^{(r)}h)$ *is a cusp form if* $r > 0$*.*

Lemma 0.5. *For arbitrary elements* $g \in \mathcal{M}_l(\Gamma_1(N))$ *and* $h \in \mathcal{M}_m(\Gamma_1(N))$*, we have*

$$H(g\delta_m^{(r)}h) = (-1)^r H(h\delta_l^{(r)}g).$$

CHAPTER 4

Distributions and measures

In this chapter, we introduce the formalism of p-adic analysis. While some of the results may initially appear somewhat abstract, mastering the measure-theoretic language will greatly simplify otherwise complex calculations. Throughout this chapter, we denote by T a locally compact, totally disconnected topological space (with additional conditions specified as needed). Additionally, let W be an abelian group.

EXAMPLE 0.1. For example T can be a Galois group of a p-adic number field K and W can be a K-Banach space.

1. Distributions

Let $\mathrm{Step}(T)$ be the group of $\mathbb{Z}$-valued functions on T that are locally constant with compact support.

EXAMPLE 1.1. For any compact subset U of T, the characteristic function χ_U belongs to $\mathrm{Step}(T)$.

DEFINITION 1.1. A W-valued **distribution** on T is a group homomorphism $\mu : \mathrm{Step}(T) \to W$. The set of all W-valued distributions on T is denoted by $\mathrm{Dist}(T) = \mathrm{Hom}(\mathrm{Step}(T), W)$. For $\phi \in \mathrm{Step}(T)$, the value of μ at ϕ is denoted by

$$\int_T \phi(t)\, d\mu := \mu(\phi).$$

Let $\mathcal{A}(T, W)$ be the set of all finitely additive W-valued functions on compact open subests of T. To give a measure theoretic interpretation, we observe that there is a bijection between $\mathrm{Dist}(T, W)$ and $\mathcal{A}(T, W)$. Explicitly,

$$\mathrm{Dist}(T, W) \to \mathcal{A}(T, W)$$
$$\mu \mapsto [U \mapsto \mu(\chi_U)].$$

We also denote by $\mu \in \mathcal{A}(T, W)$ the corresponding element under this bijection. Conversely, given a compact open subset $U \subset T$,

$$\mu(U) = \int_U d\mu := \mu(\chi_U) = \int_T \chi_U d\mu.$$

1.1. Distributions on (pro)finite sets. Suppose that T is finite. In this case, the singleton subsets of T are both compact and open, and every compact open set is a union of these singletons. This implies that

$$\mathcal{A}(T, W) \simeq \text{the abelian group of } W\text{-valued functions on } T.$$

Thus, when T is finite, we can identify $\mathrm{Dist}(T, W)$ with the abelian group of W-valued functions on T.

Now suppose that T is profinite, i.e., T is the inverse limit of a collection of finite sets T_i indexed by a directed poset I. For $i \geq j$ in I, there are surjections

$$\pi_{ij} : T_i \twoheadrightarrow T_j,$$

such that for $i \geq j \geq k$, we have $\pi_{jk} \circ \pi_{ij} = \pi_{ik}$. The notion of a W-valued distribution on T can then be reformulated as a collection of W-valued maps:

$$\mu_j : T_j \to W,$$

satisfying the condition

$$\mu_j(x) = \sum_{\{y \in T_i : \pi_{ij}(y) = x\}} \mu_i(y), \quad \text{for } i \geq j \text{ and } x \in T_j. \tag{7}$$

We know that $\mathrm{Dist}(T_i, W)$ is the abelian group of W-valued functions on T_i for all $i \in I$.

For $i \geq j$, define

$$\begin{aligned} N_{ij} : \mathrm{Dist}(T_i, W) &\to \mathrm{Dist}(T_j, W) \\ \mu_i &\mapsto \mu_j := N_{ij}(\mu_i), \end{aligned}$$

where

$$\mu_j(x) = \sum_{\{y \in T_i : \pi_{ij}(y) = x\}} \mu_i(y), \quad \text{for } i \geq j \text{ and } x \in T_j.$$

We claim that

$$\mathrm{Dist}(T, W) \simeq \varprojlim_i \mathrm{Dist}(T_i, W).$$

Let $\mu \in \mathrm{Dist}(T, W)$ and $t_i \in T_i$. Define

$$\begin{aligned} \chi_{i,t_i} : T &\to \mathbb{Z} \\ t &\mapsto \chi_{\{t_i\}}(\pi_i(t)). \end{aligned}$$

Since the set of all elements $t \in T$ such that $\pi_i(t) = t_i$ is compact and open, it follows that $\chi_{i,t_i} \in \mathrm{Step}(T)$. We define $\mu_i \in \mathrm{Dist}(T_i, W)$ by

$$\mu_i(t_i) = \mu(\chi_{i,t_i}).$$

By construction, the collection $\{\mu_i(x)\}_{i \in I}$ satisfies equation (7). Conversely, given a system $\{\mu_i\}_{i \in I}$ that satisfies equation (7), this system corresponds to a unique distribution $\mu \in \mathrm{Dist}(T, W)$.

2. Measures

We now suppose that W is a finite-dimensional Banach space over an extension K of $\mathbb{Q}_p$, as this case is of the most importance to us.

DEFINITION 2.1. A W-valued measure μ on T is a bounded W-valued distribution μ on T. If a measure μ takes values in a subgroup $A \subset W$, then we call μ an A-valued measure.

For any two given topological spaces X and Y, we denote by $\mathcal{C}(X,Y)$ the space of continuous maps from X to Y. We can define a norm on $\mathcal{C}(T,K)$ by

$$|\phi| = \sup_{g\in G} \left|\phi(g)\right|_p, \quad \phi \in \mathcal{C}(T,K).$$

This norm makes $\mathcal{C}(T,K)$ into a complete topological K-vector space whose topology is defined by a norm $|\cdot|$ satisfying:

(1) $|f| = 0$ if and only if $f = 0$;

(2) $|f+g| \leq \max(|f|,|g|)$ for all $f, g \in \mathcal{C}(T,W)$;

(3) $|af| = |a|_p|f|$ for all $a \in A$ and $f \in \mathcal{C}(T,W)$.

A linear map of K-Banach spaces ψ on $\mathcal{C}(T,K)$ into W is a called **bounded** if there is a positive constant B such that $\left|\psi(\phi)\right|_W \leq B|\phi|$ for all $\phi \in \mathcal{C}(T,K)$.

PROPOSITION 2.1. *A W-valued measure μ : Step$(T) \to W$ on T extends to a unique bounded homomorphism of K-banach spaces*

$$\begin{aligned} &\mathcal{C}(T,K) \to W, \\ &f \mapsto \int_T f d\mu. \end{aligned} \tag{8}$$

This gives a one-to-one correspondence:

$$\boxed{W\text{-valued measures on } T} \leftrightarrow \boxed{\begin{array}{c}\text{bounded homomorphisms of the}\\ K\text{-Banach spaces: } \mathcal{C}(T,K) \to W\end{array}}.$$

PROOF. See [**MSD74**] page 37-38. ∎

REMARK. From the proof in [**MSD74**] it is clear that instead of the field K we can take any closed subring of $\mathbb{C}_p$.

2.1. Measures on profinite groups. Given a profinite group $T = \varprojlim_i T_i$, we know that a W-valued distribution on T is equivalent to a system $\{\mu_i\}_{i\in I}$ of W-valued functions μ_i on T_i satisfying the equation 7, using the bijection:

$$\mathrm{Dist}(T,W) \simeq \varprojlim_i \mathrm{Dist}(T_i,W).$$

Thus a measure on T is equivalent to a system $\{\mu_i\}_{i\in I}$ of bounded W-valued functions μ_i on T_i satisfying equation 7. In what follows, we will construct some important types of measures on profinite groups $T = \varprojlim_i T_i$ using this idea, i.e., we construct a measure on T by constructing W-valued bounded functions on T_i (W will be the space of p-adic modular forms) which satisfy the compatibility condition of equation 7.

3. Measures with values in modular forms

3.1. Eisenstein measure. Kummer proved the following congruences of the values of the Riemann zeta functions at the negative integers. He proved that if k and k' are positive even integers not divisible by $p-1$, then for all positive integers d,

$$(1-p^{k-1})\zeta(1-k) \equiv (1-p^{k'-1})\zeta(1-k') \pmod{p^d}$$

whenever $k \equiv k' \pmod{\varphi(p^d)}$, with φ denoting the Euler's totient function.

Question 1. Given an L-function whose values at certain points are known to be algebrais, how mighr we construct a p-adic L-functions encoding congruences between values of (a suitably modified at p) version of that L-function?

Question 2. How might we construct p-adic families of Eisenstein series or, more specifically, p-adic Eisenstein measures?

Example 3.1. Let $k \geq 4$ be an even integer. Consider the level 1, weight k Eisenstein series G_k whose Fourier expansion is given by

$$G_k(z) = \frac{\zeta(1-k)}{2} + \sum_{n\geq 1} \sigma_{k-1}(n)q^n$$

where $q = e^{2\pi i z}$ and $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$. In the 1800s, Kummer proved that if $p-1 \nmid k$, then $\frac{\zeta(1-k)}{2}$ is p-integral. Additionally, he showed that if $k \equiv k' \pmod{p-1}$, then

$$\frac{\zeta(1-k)}{2} \equiv \frac{\zeta(1-k')}{2} \pmod{p}$$

(see [?]). By applying Fermat's little theorem to the non-constant coefficients, we also find that

$$G_k \equiv G_{k'} \pmod{p}$$

whenever $k \equiv k' \not\equiv 0 \pmod{p-1}$.

Let $M \in \mathbb{Z}_{>0}$ and put

$$Z_v = (\mathbb{Z}/Mp^v\mathbb{Z})^\times \quad \text{and} \quad Z = \varprojlim_v Z_v.$$

We define a measure on the space Z with values in p-adic modular forms. Precisely, the values are p-adic Eisenstein series, thus we call it p-adic Eisenstein measure.

For each $v \geq 0$ and $a \in Z_v$ we define the following Eisentein series:

$$E_{m,v}(a) = \zeta(1-m; a, Mp^v) + \sum_{n=1}^{\infty} \Big(\sum_{\substack{d \mid n \\ d \equiv a (Mp^v)}} \operatorname{sgn}(d) d^{m-1} \Big) q^n,$$

where

$$\zeta(s; a, Mp^v) = \sum_{0 < n \equiv a (Mp^v)} n^{-s}$$

is the partial zeta function modulo Mp^v. It is well known that $E_{m,v}(a)$ is a modular form of weight m for $\Gamma_1(Mp^v)$ with rational coefficients if $Mp^v > 2$. It is clear from the definition that the coefficients of the q-expansion of $E_{m,v}(a)$ are integral except the constant term.

Proposition 3.1. *The p-adic norms* $\left|E_{m,v}(a)\right|_p$ *(varying* $v \geq 0$ *and* $a \in Z_v$*) in* $\overline{\mathcal{M}}(M; \mathbb{Q}_p)$ *are not bounded.*

Proof. We only need to focus on the constant term $\zeta(1-m; a, Mp^v)$ of $E_{m,v}$. It is known that

$$\zeta(1-m; a, Mp^v) = (Mp^v)^{m-1} \sum_{i=0}^{m} B_i \left(\frac{a}{Mp^v} \right)^{m-i,}$$

where B_i is the i-th Bernoulli's number. ∎

It turns out that the system $\{E_{m,v}(a)\}_{v \geq 2, a \in Z_v}$ satisfies the condition of equation 7. Thus, we get a distribution on $\mathbb{Z}$ with values in $\overline{\mathcal{M}}(M; \mathbb{Q}_p)$. To get a measure we need to tweek our this Eisenstein series: for any integer $b \geq 1$ prime to Mp, define:

$$E^b_{m,v}(a) = E_{m,v}(a) - b^m E_{m,v}(b^{-1}a), \tag{9}$$

where we take the inverse b^{-1} in Z_v considering b to be an element of Z_v naturally.

Proposition 3.2. *The series* $E^b_{m,v}(a)$ *is p-adically integral.*

PROOF. Fix $0 \le c < Mp^v$ such that $bc \equiv a \pmod{Mp^v}$. Then,

$$\zeta(1-m;a,Mp^v) - b^m\zeta(1-m;c,Mp^v)$$
$$= (Mp^v)^{m-1}\sum_{i=0}^{m} B_i \left(\left(\frac{a}{Mp^v}\right)^{m-i} - b^m \left(\frac{c}{Mp^v}\right)^{m-i} \right).$$

For the first term in this summation we have

$$(Mp^v)^{m-1}B_0\left[\left(\frac{a}{Mp^v}\right)^{m} - b^m\left(\frac{c}{Mp^v}\right)^{m}\right] = \frac{1}{Mp^v}B_0(a^m-(bc)^m),$$

which is clearly p-adically integral. Other terms are also p-adically integral because the denominator of B_i is divisible by p at the most. ∎

As a result of the above proof, if we define another system

$$\varepsilon^b_{m,v}(a) = \zeta(1-m;a,Mp^v) - b^m\zeta(1-m;b^{-1}a,Mp^v) \in \mathbb{Q}. \tag{10}$$

Then, we know that $\left|\varepsilon^b_{m,v}(a)\right|_p$ is bounded independently of $a \in Z_v$ and $v \ge 0$. Furthermore, similar to the system $\{E_{m,v}(a)\}_{v\ge 2, a\in Z_v}$, the system $\{\varepsilon_{m,v}(a)\}_{v\ge 2, a\in Z_v}$ also satisfies the compatibility condition 7. Thus the systems 9 and 10 for each positive integer m gives bounded measure on Z with values in $\overline{\mathcal{M}}(M;\mathbb{Q}_p)$ and $\mathbb{Q}_p$, respectively. We denote the respective measures by E^b_m and ε^b_m.

3.2. Theta Measure. Let V be a $\mathbb{Q}$-vector space having even dimension 2κ, endowed with a quadratic form $q: V \to \mathbb{Q}$ satisfying:

(1) $q(av) = a^2q(v)$ for all $a \in \mathbb{Q}$ and $v \in V$;

(2) and the function $b(u,v) := q(u+v) - q(u) - q(v)$ is bilinear.

We note that $q(v) = \frac{1}{2}b(v,v)$. Furthermore, we suppose that q is positive definite: $q(v) > 0$ for $0 \ne v \in V$. Let $L \subset V$ be an integral lattice of V with respect to q i.e., $q(L) \subset \mathbb{Z}$, so that $b(v,v) \in 2\mathbb{Z}$ and $b(u,v) \in \mathbb{Z}$ for all $u, v \in L$. The *dual lattice* of L with respect to q is defined by

$$L^* := \{v \in V : b(v,L) \subset \mathbb{Z}\}.$$

Since $q(L) \subset \mathbb{Z}$, we have $L \subset L^*$. The quotient L^*/L is a finite abelain group and is called the *discriminant group* of L. The *level* of a lattice L is the least positive integer M such that $Mq(L^*) \subset \mathbb{Z}$. We also observe that M annihilates L^*/L.

We choose a basis $\{v_i\}$ of V such that $L = \oplus_i \mathbb{Z}v_i$. Since, L is integral, the matrix A of q with respect to the basis $\{v_i\}$ is integral and positive definite.

We now define the differential operator Δ_A by

$$\Delta_A = \sum_{i,j=1}^{r} b_{ij}\delta^2/\delta x_i \delta x_j, \quad A^{-1} = (b_{ij}).$$

DEFINITION 3.1. Let $P(x)$ be a homogeneous polynomial of degree α with complex coefficients in variables $x_1, \dots, x_{2\kappa}$. We call $P(x)$ a **spherial function** of degree α with respect to A if

$$\Delta_A P(x) = 0.$$

Using the basis $\{v_i\}$, we can identify V with $\mathbb{C}^{2\kappa}$, thus we can consider P to be a function $P : V \to \mathbb{C}$. It is well known [**Iwa97**] that any spherical function of degree α is given by:

(1) either P is a constant,

(2) $P(v) = b(w, v)$ for some $w \in V \otimes_{\mathbb{Q}} \mathbb{C}$,

(3) or P can be expressed as follows: there exist finitely many $w \in V \otimes_{\mathbb{Q}} \mathbb{C}$ with $q(w) = 0$ such that

$$\eta(v) = \sum_{w} c(w) b(w, v)^\alpha,$$

where $c(w) \in \mathbb{C}$ and α is an integer ≥ 2.

We refer to [**Iwa97**] chapter 9 for more details on spherical functions.

DEFINITION 3.2. Given a $\mathbb{C}$-valued function $h : L^* \to \mathbb{C}$, we *formally* define the associated *theta series* as:

$$\Theta(h)(z) = \sum_{v \in L^*} h(v) e^{2\pi i q(v) z}.$$

Let $\Phi : L^*/L \to \mathbb{C}$ be a function. If $h = \Phi\eta$ then this series converges, and defines a holomorphic function on $\mathbb{H}$. The main aim of this subsection is to determine the behavior of Θ under the action of $\Gamma_0(M)$ (acting on the variable z). To write the transformational formula with ease we define an action of $\Gamma_0(M)$ on the set of all functions $\Phi : L^*/L \to \mathbb{C}$ via

$$(\gamma \cdot \Phi)(v) = e^{2\pi i d b q(v)} \Phi(dv), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(M).$$

We have the following transformation formula of theta functions.

PROPOSITION 3.3. *The function $\Theta(\Phi\eta)(z)$ satisfies the transformation formula:*

$$\Theta(\Phi\eta)(\gamma \cdot z) = \left(\frac{\Delta}{d}\right) \left(\frac{2c}{d}\right)^{2\kappa} \varepsilon_d^{-2\kappa} (cz + d)^{\kappa+\alpha} \Theta((\gamma \cdot \Phi)\eta)(z), \tag{11}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(M)$, $\left(\frac{\Delta}{d}\right)$, $\left(\frac{2c}{d}\right)$ *are Jacobi symbols, and*

$$\varepsilon_d = \begin{cases} 1 & \text{if } d \not\equiv 3 \pmod 4 \\ i & \text{if } d \equiv 3 \pmod 4. \end{cases}$$

3.2.1. *Theta functions as modular forms.* For each integer $n \geq 0$, $p^n L$ is again a lattice. Recall that the level M of L is the smallest positive integer such that $Mq(L^*) \subset \mathbb{Z}$. Since $(p^n L)^* = p^{-n} L^*$ and $q(p^{-n}L^*) = p^{-2n} q(L^*)$, we conclude that the level of $p^n L$ is Mp^{2n}. Given that $p^n L \subset L \subset L^*$, we can define:

$$X := \varprojlim_{v} L^*/p^n L.$$

Furthermore, let $\mathcal{W} = \{v \in L^* : q(v) \in \mathbb{Z}\}$, and for each positive integer n put $\mathcal{W}_n = W/p^n L$. Since $L \subset \mathcal{W}$, the following inverse limit is also well-defined:

$$W := \varprojlim_{n} \mathcal{W}_n.$$

The inclusion $\mathcal{W} \subset L^*$ induces an injection $W \hookrightarrow X$ using the universal property of inverse limits. Additionally, we observe that the quadratic form q naturally extends to $q : W \to \mathbb{Z}_p$. Indeed, since

$$q(p^n L) = p^{2n} q(L) \subset p^{2n}\mathbb{Z} \subset p^n \mathbb{Z},$$

the map

$$q : \mathcal{W} \to \mathbb{Z} \twoheadrightarrow \mathbb{Z}/p^n\mathbb{Z}$$

factors through $q : \mathcal{W}_n \to \mathbb{Z}/p^n\mathbb{Z}$. Therefore, by the universal property of inverse limits, we can uniquely extend q to $q : W \to \mathbb{Z}_p$.

Let $\eta : V \to \overline{\mathbb{Q}}$ be any spherical function on V of degree $\alpha \geq 0$ which takes algebraic values. Since, $\mathcal{W}$ is dense in W, we can extend η to $\eta : W \to \overline{\mathbb{Q}}$, and composing it with the fix embedding $i : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$, we obtain a function from $W \to \mathbb{C}_p$, which is again denoted by η. Fix a finite extension K of $\mathbb{Q}_p$ so that η has values in K. For each $w \in \mathcal{W}_n$, put

$$\Theta_n(w, \eta) := \sum_{\substack{v \in \mathcal{W} \\ v \equiv w \pmod{p^n L}}} \eta(v) e(q(v)z) \tag{12}$$

LEMMA 3.1. $\Theta_n(w, \eta) \in \mathcal{M}_{\kappa+\alpha}(\Gamma_1(Mp^{2n}); K)$.

PROOF. Follows from the transformational formula of theta series. Details to be written. ■

The system, $\{\Theta_n(w, \eta)\}_{n \geq 0, w \in W_n}$ defines a measure on W with values in the K-Banach space $\overline{\mathcal{M}}(M; K)$.

(1) When η is the constant function with value 1 on V, this measure will be called the theta measure attached to the quadratic space V, and is denoted by Θ or $d\Theta$.

(2) For any continuous function $\phi \in \mathcal{C}(W;K)$, the value

$$\Theta(\phi) := \int_W \phi d\Theta = \sum_{w\in\mathcal{W}} \phi(w)q^{q(w)} \in \overline{\mathcal{M}}(M;K).$$

(3) Thus, it is clear that the measure attached to the system $\{\Theta_n(w,\eta)\}_{n\geq 0, w\in W_n}$ is the product measure $\eta\cdot d\theta$.

3.3. Convolution of Eisenstein and theta measure. In this paragraph we construct another measure as a "convolution" of the standard measures that we have constructed above, namely, Eisenstein and theta measure.

We have the natural action of $(\mathbb{Z}/Mp^n/Z)^\times$ on L^*/p^nL, and the subset W_n of L^*/p^nL is stable under this action of Z_n. Thus, we can consider $\Theta_n(a\cdot w,\eta)$ for $w\in W_n$ and $a\in Z_n$. Let ω be a Dirichlet character modulo Mp^n. For each non-negative integer r and each positive integer m, we define

(13)

$$\begin{aligned}\Phi_n(w) &= \Phi_n(w;r,m,\omega,\eta)\\ &= \sum_{a\in Z_n} \omega(a)H[\theta_n(a\cdot w,\eta)\delta_m^r E_{m,n}^b(a)] \in \mathcal{M}k(\Gamma_1(Mp^{2n});K),\end{aligned}$$

where $k=\kappa+\alpha+m/2r$, δ_m^r is Shimura's diffrential operator defined in chapter 3, and H is the holomorphic projection map, which we also defined in chapter 3.

THEOREM 3.2. *The system* $\{\Phi_n(w)\}_{\substack{n\geq 2\\ w\in W_n}}$ *defines a measure on* W *with values in* $\overline{\mathcal{M}}_k(M;K)$. *We denote this measure by* $\Phi=\Phi(r,m,\omega,\eta)$.

PROOF. It follows from lemma 0.3 that $\left|\Phi_n(w)\right|_p$ is bounded independently of n and $w\in W_n$. Now we need to show that the system $\{\Phi_n(w)\}_{\substack{n\geq 2\\ w\in W_n}}$ satisfy the compatibility condition 7. Let $i\geq j\geq 2$, then

$$\sum_{\substack{w\in W \\ w\equiv x \pmod{p^j L}}} \Phi_i(w) = \sum_{a\in Z_i} \omega(a) H[\big(\sum_{\substack{w\in W_i \\ w\equiv x \pmod{p^j L}}} \theta_i(a\cdot w,\eta)\big)\delta_m^r E_{m,i}^b(a)]$$

$$= \sum_{a\in Z_i} \omega(a) H[\theta_j(a\cdot x,\eta)\delta_m^r E_{m,i}^b(a)]$$

$$= \sum_{a\in Z_j} \omega(a) H[\theta_j(a\cdot x,\eta)\delta_m^r \big(\sum_{\substack{c\in Z_i \\ c\equiv a \pmod{Mp^j}}} E_{m,j}^b(a)\big)]$$

$$= \sum_{a\in Z_j} \omega(a) H[\theta_j(a\cdot x,\eta)\delta_m^r E_{m,j}^b(a)] = \Phi_j(x).$$

■

LEMMA 3.3. *Let γ be an element of $\Gamma_0(Mp^n)$ such that $\gamma \equiv \begin{pmatrix} t & * \\ 0 & t^{-1}\end{pmatrix}$ $\pmod{Mp^n}$. Then, we have*

$$E_{m,n}(a)[\gamma]_m = E_{m,n}(at), \text{ if } Mp^n > 2.$$

PROOF. See Lemma 6.1 in [**Hid85**]. ■

It is clear from Proposition 3.3 that, for $\gamma \in \Gamma_0(Mp^{2n})$ with $\gamma \equiv \begin{pmatrix} t & * \\ 0 & t^{-1}\end{pmatrix}$ $\pmod{Mp^{2n}}$, we have the following:

$$\theta_n(w,\eta)[\gamma]_{\kappa+\alpha} = \chi_0(t)\theta_n(tw,\eta),$$

where

$$\chi_0(a) = \left(\frac{(-1)^\kappa \Delta}{a}\right), \text{for } \Delta = [L^* : L]$$

is the Legendre symbol. Then the above lemma shows that $\Phi_n(w) \in \mathcal{M}_k(\Gamma_0(Mp^{2v}), \omega\chi_0; K)$ for $k = \kappa+\alpha+m+2r$. Thus

(14)

$\Phi(r,m,\omega,\eta)$ has values in $\overline{\mathcal{M}}_k(M,\omega\chi_0;K)$ for $k = \kappa+\alpha+m+2r$.

We shall now define a measure that has values in the space of ordinary forms. For any $\mathcal{C}(W;K)$, the value $\theta(\phi) = \sum_{w\in\mathcal{W}} \phi(w)e^{2\pi i q(w)}$ is an element of $\overline{\mathcal{M}}(M;K)$. Then, it is plain that, for $0 \le r \in \mathbb{Z}$,

$$d^r\theta(\phi) = \theta(q^r\phi),$$

where d us the differential operator defined in Chapyter 3. It is known that d takes $\overline{\mathcal{M}}(M;K)$ into itself [**Kat76**]. We extend the $T(p)$ Hecke operator to $K[[q]]$ by

$$T(p)\left(\sum_{n=0}^\infty a(n)e^{2\pi i n}\right) = \sum_{n=0}^\infty a(np)e^{2\pi i n},$$

and put

$$\left|\sum_{n=0}^{\infty} a(n)e^{2\pi i n}\right|_p = \sup_n \left|a(n)\right|_p.$$

Then, we can define the valuation ring by

$$\mathcal{U} = \{F \in K[[q]] : |F|_p \text{ is finite }\}.$$

LEMMA 3.4. *The valuation ring $\mathcal{U}$ is stable under the differential operator d and the Hecke operator $T(p)$. Furthermore,*

$$\lim_{m\to\infty} T(p)^m(dF) = 0 \textit{ if } F \in \mathcal{U}.$$

The space $\overline{\mathcal{M}}(M;K)$ can be regarded as a subspace of $\mathcal{U}$ by considering the q-expansion. Then, we can naturally extend the definition of the Hida's idempotent e of $\mathcal{H}(M;\mathcal{O}_K)$ as follows:

LEMMA 3.5. *The idempotent e can be naturally extended to an operator on $d\mathcal{U} + \overline{\mathcal{M}}(M;K)$ so that e annihilates $d\mathcal{U}$.*

This allows us to define the ordinary part $\Phi^O = \Phi^O(r,m,\omega,\eta)$ of the measure $\Phi(r,m\omega,\eta)$ as follows:

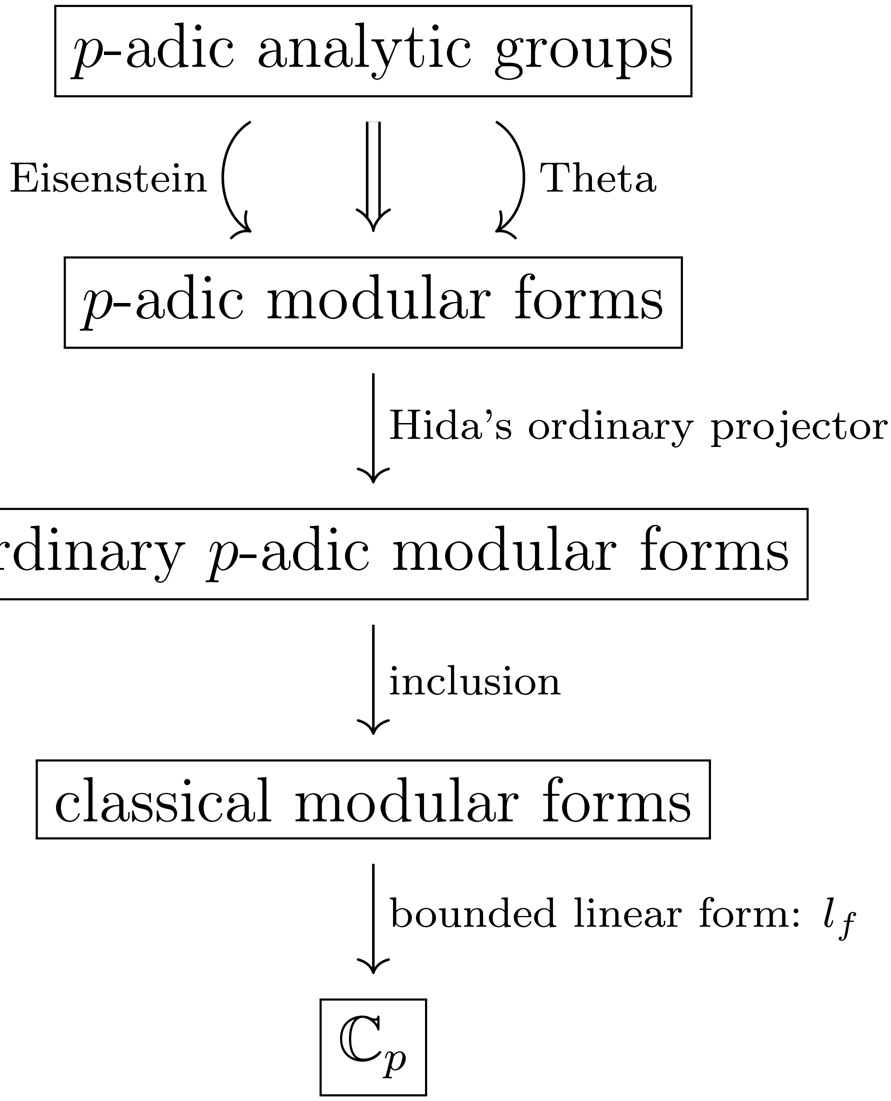
$$\int_W \phi d\Phi^O(r,m,\omega,\eta) = e[\int_W \phi d\Phi(r,m,\omega,\eta)] \text{ for } \phi \in \mathcal{C}(W;K). \tag{15}$$

It is clear to see that the measure Φ^O has values in the finite dimensional K-vector space $\overline{\mathcal{M}}_k^O(M,\omega\chi_0;K)$ (from Theorem 3.2).

THEOREM 3.6. *Let k and r be integers with $k > \kappa$ and $0 \le r < \frac{k-\kappa}{2}$, and assume that the degree α of η is less than $k-\alpha-2r$. Then, we have*

$$\Phi^O(r,k-\kappa-\alpha-2r,\omega,\eta) = (-1)^r \eta q^r \cdot \Phi^O(0,k-\kappa,\omega,1).$$

PROOF. See Proposition 6.3 in [**Hid85**]. ∎


p-adic analytic groups
Eisenstein
Theta
p-adic modular forms
Hida's ordinary projector
ordinary p-adic modular forms
inclusion
classical modular forms
bounded linear form: l_f
$\mathbb{C}_p$

CHAPTER 5

Hida's contruction of a p-adic measure

Having all the machinery at our disposal we now present a slightly weaker version of the main Theorem 5.1 and demonstrate its proof.

Firstly, we need to recall a bit of the notations from the previous chapters. Let V be a quadratic $\mathbb{Q}$-vector space with quadratic form q. We suppose that dimension of V is even 2κ, where κ is an integer. Let L be an integral lattice in V of level M, i.e., $q(L) \subset \mathbb{Z}$, and L^* be its dual lattice. Let

$$X := \varprojlim_{v} L^*/p^n L.$$

Furthermore, let $\mathcal{W} = \{v \in L^* : q(v) \in \mathbb{Z}\}$, and for each positive integer n put $\mathcal{W}_n = W/p^n L$. Since $L \subset \mathcal{W}$, the following inverse limit is also well-defined:

$$W := \varprojlim_{n} \mathcal{W}_n.$$

The inclusion $\mathcal{W} \subset L^*$ induces an injection $W \hookrightarrow X$ using the universal property of inverse limits. Additionally, we observe that the quadratic form q naturally extends to $q : W \to \mathbb{Z}_p$.

Let $\eta : V \to \overline{\mathbb{Q}}$ be any spherical function on V of degree $\alpha \geq 0$ which takes algebraic values. Since, $\mathcal{W}$ is dense in W, we can extend η to $\eta : W \to \overline{\mathbb{Q}}$, and composing it with the fix embedding $i : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$, we obtain a function from $W \to \mathbb{C}_p$, which is again denoted by η. Note that the group

$$Z = \varprojlim_{v} (/Z/Mp^v\mathbb{Z})^\times$$

has a natural action on the space X, which leaves W stable. Let $\phi : W \to \overline{\mathbb{Q}}$ be an arbitrary locally constant function on Y with the property that there is a character χ of finite order of the group such that

$$\phi(zw) = \chi(z)\phi(w) \quad \text{for all } z \in Z \text{ and } w \in W. \tag{16}$$

We then define the following θ-series:

$$\theta(\phi\eta) = \sum_{w \in \mathcal{W}} \phi(w)\eta(w)e^{2\pi i q(w)z}.$$

Put

$$\xi(a) = \chi(a)\left(\frac{\Delta}{a}\right)\left(\frac{-1}{a}\right)^{\kappa},$$

where the symbols on the right hand side are the Legendre symbols, and $\Delta = [L^* : L]$. We know from Proposition 3.3 that there exists $\beta \geq 0$ such that the conductor of ξ divides Mp^{β} and $\theta(\phi\eta)$ belongs $\mathcal{M}_{\kappa+\alpha}(\Gamma_0(Mp^{\beta}), \xi)$. In the following, β will denote any fixed integer satisfying this property with $\beta \geq 1$.

As in the introduction, let f be a fixed primitive form of conductor C, with character ψ (mod C) and of weight $k \geq 2$. Assume that the p-th fourier coefficient $a(p, f)$ of f is a unit in $\mathbb{C}_p$. Let f_0 be the ordinary form associated with f and write C_0 for teh smallest possible level of f_0. Let V be a $\mathbb{Q}$-vector space having even dimension 2κ, endowed with a quadratic form q. Let L be a lattice of V of level M. Let $\mu \geq 1$ and $\lambda \geq 0$ be defined as follows:

$$C_0 = C'p^{\mu}, \quad M = M'p^{\lambda}, \quad (C', p) = (M', p) = 1.$$

We assume that C' divides M', then the following holds:

$$\mathcal{M}_k(\Gamma_0(C_0), \psi) \subseteq \mathcal{M}_k(\Gamma_0(Mp^{\mu-\lambda}), \psi),$$

this holds because $Mp^{\mu-\lambda}/C_0 = M'/C'$ which is an integer. Let W be as in §§ 3.2 and $\eta : V \to \overline{\mathbb{Q}}$ be an arbitrary spherical function on V of degree $\alpha \geq 0$.

Remark. This assumption is not a very strong assumption, since it can always be achieved by replacing L be a suitable sub-lattice.

Let us also write

$$\gamma = \begin{pmatrix} M'/C' & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \tau_{\beta} = \begin{pmatrix} 0 & -1 \\ Mp^{\beta} & 0 \end{pmatrix}.$$

Theorem 0.1. *For each integer $b > 1$, with $(b, Mp) = 1$, there exists a unique bounded measure φ_b on W with values in $\mathbb{C}_p$ satisfying the following interpolation property: for each non-negative integer r with $0 \leq 2r + \alpha < k - \kappa$, we let $j = \kappa + \alpha + 2r$, and we have that the value of the p-adic integral*

$$\int_W \phi\eta q^r \, d\varphi_b$$

is given by the image under $\iota : \overline{\mathbb{Q}} \to \mathbb{C}_p$ of

$$(17) \quad t(1 - b^{k-j}\psi\overline{\xi}(b))a(p, f_0)^{\mu-\lambda}\frac{L_{Mp^{\beta}}(f_0[\gamma]_k \otimes \theta(\phi\eta)[\tau_{\beta}]_{\kappa+\alpha}, j - r)}{\pi^{j+1}\langle h, f_0\rangle_{C_0}},$$

where $a(p, f_0)$ *is the* p*-th Fourier coefficient of* f_0*,*

$$h = f_0^\rho \left[\begin{pmatrix} 0 & -1 \\ C_0 & 0 \end{pmatrix} \right]_k ,$$

and

$$t = t(r, \alpha, \beta) = 2^{1-k-j} i^{k+j} p^{(\mu-\lambda)(1-k/2)+\beta j/2} (M)^{(j-k)/2+1} \Gamma(j-r)\Gamma(r+1).$$

Remark. (1) It is easy to see that 17 is independent of the choice of β.

(2) The uniqueness of φ_b follows from the fact that any locally constant function on W is a finite sum of those satisfying 16.

(3) Finally, it should be noted that the above theorem does not give the p-adic interpolation at all of the special values $L_{Mp^\beta}(f_0[\gamma]_k \otimes \theta(\phi\eta)[\tau_\beta]_{\kappa+\alpha}, m)$ with m satisfying $\kappa + \alpha \leq m < k$, where algebraicity is known.

We first construct the measure φ_b as in the Theorem 0.1 for each $b > 1$ prime to Mp. Let K be a sufficiently large finite extension of $\mathbb{Q}_p$ which contains all the fourier coefficients of f_0. Let χ_0 be the Dirichlet character modulo M define as:

$$\chi_0(a) = \left(\frac{(-1)^\kappa \Delta}{a} \right), \text{for } \Delta = [L^* : L].$$

Let $\Phi^0 = \Phi^0(0, k - \kappa, \psi\chi_0, 1)$ be the bounded measure on $\mathcal{C}(W; K)$ defined in Equation 13. Then, the measure has values in the space $\overline{\mathcal{M}}_k^O(M, \psi; K)$, which is a subspace of $\mathcal{M}_k(\Gamma_0(Mp^{\mu-\lambda}), \psi; \mathbb{C}_p)$.

We now define a trace operator:

$$\begin{aligned} \mathrm{Tr} : \mathcal{M}_k(\Gamma_0(Mp^{\mu-\lambda}), \psi; \mathbb{C}_p) &\to \mathcal{M}_k(\Gamma_0(C_0), \psi; \mathbb{C}_p) \\ g &\mapsto \sum_\gamma \overline{\psi}(\gamma) g[\gamma]_k, \end{aligned} \tag{18}$$

where γ runs over a representative set for $\Gamma_0(Mp^{\mu-\lambda})\backslash\Gamma_0(C_0)$ and

$$\overline{\psi}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \overline{\psi(d)}.$$

Lemma 0.2. *The linear operator*

$$\mathrm{Tr} : \mathcal{M}_k(\Gamma_0(Mp^{\mu-\lambda}), \psi; \mathbb{C}_p) \to \mathcal{M}_k(\Gamma_0(C_0), \psi; \mathbb{C}_p)$$

is a bounded linear operator.

We choose K such that the trace operator Tr sends $\mathcal{M}_k(\Gamma_0(Mp^{\mu-\lambda}),\psi;K)$ onto $\mathcal{M}_k(\Gamma_0(C_0),\psi;K)$.

We define the measure φ_b by

$$\int_W \phi d\varphi_b = l_f\left[\mathrm{Tr}\left(\int_W \phi d\Phi^0\right)\right] \text{ for } \phi\in\mathcal{C}(W;K), \tag{19}$$

where $l_f : \overline{\mathcal{M}}(C_0,\psi;K)\to K$ is the bounded linear form associted with f as explained in § 4.

Let us now assume that the degree α of η is less than $k-\kappa$, and ϕ be a locally constant function on W with algebraic values such that:

$$\phi(aw)=\chi(a)\phi(w) \text{ for every } a\in Z, w\in W,$$

where χ is a character of Z of finite order. Put,

$$\xi(a)=\chi(a)\chi_0(a) \text{ for } a\in\mathbb{Z} \text{ prime to } Mp.$$

Then, for sufficiently large $\beta\geq 1$, the theta series $\theta(\phi\eta)$ belongs to $\mathcal{M}_{\kappa+\alpha}(\Gamma_0(Mp^\beta),\xi)$ (from Proposition 3.3). Let us fix such a β throughout. Let r be an arbitrary integer with $0\leq 2r+\alpha<k-\kappa$. We shall now evaluate the integral $\int_W \phi\eta q^r d\varphi_b$ as in Theorem 0.1. We assume that η and ϕ have values in K using the fixed embedding ι.

Take a positive integer n so that ϕ factors through $W_n=\mathcal{W}/p^nL$. We may assume that $n\geq\beta$ and $n>\mu-\lambda$. Then, Theorem 3.6 shows

$$(-1)^r\int_W \phi\eta q^r d\Phi^O = e\left[\sum_{w\in W_n}\phi(w)\cdot\Phi_n(w;r,m,\psi\chi_0,\eta)\right],$$

where $m=k-\kappa-\alpha-2r$ and $\Phi_n(w)$ is as in 13. Furthermore, from Equation 13 it follows that:

$$\begin{aligned}
&\sum_{w\in W_n}\phi(w)\Phi_n(w;r,m,\psi\chi_0,\eta)\\
&\qquad=\sum_{w\in W_n}\phi(w)\sum_{a\in Z_n}\psi\chi_0(a)H[\theta_n(aw,\eta)\delta_m^r E_{m,n}^b(a)]\\
&\qquad=\sum_a\psi\chi_0(a)\sum_w\phi(a^{-1}w)H[\theta_n(w,\eta)\delta_m^r E_{m,n}^b(a)]\\
&\qquad=\sum_a\psi\chi_0\overline{\chi}(a)H[\sum_w\phi(w)\theta_n(w,\eta)\delta_m^r E_{m,n}^b(a)]\\
&\qquad=H[\theta(\phi\eta)\delta_m^r(\sum_a\psi\overline{\xi}(a)E_{m,n}^b(a))].
\end{aligned}\tag{20}$$

Note that

$$E_{m,Mp^\beta}(\psi\overline{\xi}) = E_{m,Mp^n}(\psi\overline{\xi}) = \frac{1}{2}\sum_{a\in\mathbb{Z}_v}\psi\overline{\xi}(a)E_{m,n}(a).$$

Then, Equation 20 simplifies to

$$2(1-b^m\psi\overline{\xi}(b))H[\theta(\phi\eta)\delta_m^r(E_{m,Mp^\beta}(\psi\overline{\xi}))].$$

We have by the definition of φ_b that

$$(-1)^r\int_W \phi\eta q^r d\varphi_b = 2(1-b^m\psi\overline{\xi}(b))l_f\left[\mathrm{Tr}\left\{e\left(H(\theta(\phi\eta)\delta_m^r E_{m,Mp^\beta}(\psi\overline{\xi}))\right)\right\}\right]. \tag{21}$$

Let R be a chosen set of representative for $\Gamma_0(Mp^\beta)\backslash\Gamma_0(C_0p^{\beta+\lambda-\mu})$. Note that $Mp^\beta = M'p^{\beta+\lambda}$, $C_0p^{\beta+\lambda-\mu} = C'p^{\beta+\lambda}$, and that C' and M' are prime to p. Therefore, the set R may be regarded as a complete representative set for $\Gamma_0(Mp^{\mu-\lambda})\backslash\Gamma_0(C_0)$. Thus, one can extend the operator Tr defined in Equation 18 to the trace operator of $\mathcal{M}_k(\Gamma_0(Mp^\beta),\psi;\mathbb{C}_p)$ onto $\mathcal{M}_k(\Gamma_0(C_0p^{\beta+\lambda-\mu}),\psi;\mathbb{C}_p)$; by putting:

$$\mathrm{Tr}(g) = \sum_{\gamma\in R}\overline{\psi}(\gamma)g[\gamma]_k, \text{ for } g\in\mathcal{M}_k(\Gamma_0(C_0p^\beta),\psi;\mathbb{C}_p).$$

Then, we see easily that

$$\mathrm{Tr}\circ T(p) = T(p)\circ\mathrm{Tr} \quad\text{and}\quad \mathrm{Tr}\circ e = e\circ\mathrm{Tr}.$$

Given that $T(p)^{\beta+\lambda-\mu}(\mathrm{Tr}(g))$ for $g\in\mathcal{M}_k(\Gamma_0(Mp^\beta),\psi;\overline{\mathbb{Q}})$ lies within $\mathcal{M}_k(\Gamma_0(C_0),\psi;\overline{\mathbb{Q}})$, we can apply Proposition 4.3 to obtain the following result:

$$\begin{aligned} l_f[\mathrm{Tr}\big(e(g)\big)] &= l_f[e(\mathrm{Tr}(g))] \\ &= a(p,f_0)^{\mu-\beta-\lambda}l_f[T(p)^{\beta+\lambda-\mu}(\mathrm{Tr}(g))] \\ &= a(p,f_0)^{\mu-\beta-\lambda_p(\beta+\lambda-\mu)(k-1)}\frac{\langle h_{\beta+\lambda-\mu},\mathrm{Tr}(g)\rangle_{C_Op^{\beta+\lambda-\mu}}}{\langle h,f_0\rangle_{C_0}}, \end{aligned}$$

where

$$h = f_0^\rho\left[\begin{pmatrix}0 & -1\\ C_0 & 0\end{pmatrix}\right]_k,$$

$$h_{\beta+\lambda-\mu}(z) = h(p^{\beta+\lambda-\mu}z).$$

For

$$\tau = \begin{pmatrix}0 & -1\\ Mp^\beta & 0\end{pmatrix},$$

we have the following:

$$\langle h_{\beta+\lambda-\mu}, \mathrm{Tr}(g)\rangle_{C_O p^{\beta+\lambda-\mu}} = \langle h_{\beta+\lambda-\mu}, g\rangle_{Mp^\beta} = \langle h_{\beta+\lambda-\mu}[\tau]_k, g[\tau]_k\rangle_{Mp^\beta}.$$

Applying these formulae to $g = \mathcal{H}[\theta(\phi\eta)\delta_m^r E^b_{m,Mp^\beta}(\psi\overline{\xi})]$, we have by the property of holomorphic projection that

$$(22)\qquad \begin{aligned} (-1)^r \int_W \phi\eta q^r d\varphi_b &= 2(1-b^m\psi\overline{\xi}(b))p^{(\beta+\lambda-\mu)(k-1)}a(p,f_0)^{\mu-\beta-\lambda} \\ &\times \frac{\langle h_{\beta+\lambda-\mu}[\tau]_k, (\theta(\phi\eta)\delta_m^r E^b_{m,Mp^\beta}(\psi\overline{\xi}))[\tau]_k\rangle_{Mp^\beta}}{\langle h, f_0\rangle_{C_0}} \end{aligned}$$

where $m = k-\kappa-\alpha-2r$.

On the other hand, Lemma 7.1 ([**Hid85**]) combined with Equation 7.1 ([**Hid85**]) shows that

$$\begin{aligned} &(\delta_m^r E_{m,Mp^\beta}(\omega))[\tau]_{k-\kappa-\alpha} \\ &\quad = Ty^{-r}\sum_{0<t|N_1} \mu(t)\omega_0(t)t^{-1}J_{k-\kappa-\alpha,N_O}(t^{-1}N_1 z, -r, \overline{\omega}_0), \end{aligned}$$

where

$$\begin{aligned} &J_{k-\kappa-\alpha,N_0}(t^{-1}N_1 z, -r, \overline{\omega}_0) \\ &\qquad = \sum_{0\neq(c,d)\in\mathbb{Z}^2} \overline{\omega}_0(c)(ct^{-1}N_1 z+d)^{-(k-\kappa-\alpha)}\left|ct^{-1}N_1 z+d\right|^{2r}, \end{aligned}$$

is an Eisentein series, $\omega = \psi\overline{\xi}$, N_0 is the conductor of ω, ω_0 is the primitive character associated with ω, $N = Mp^\beta = N_0\cdot N_1$, and

$$T = N_0^{-1}G(\omega_0)\pi^{-\pi-r}2^{2r-m-1}(\sqrt{-1})^{2r-m}(Mp^\beta)^{m/2}\Gamma(m+r),$$

for $m = k-\kappa-\alpha-2r$. Note that

$$p^{(\beta+\lambda-\mu)(k-1)}h_{\beta+\lambda-\mu}[\tau]_k = (-1)^k p^{(\beta+\lambda-\mu)(k/2-1)} f_0^\rho[\gamma]_k$$

for $\gamma = \begin{pmatrix} M'/C' & 0 \\ 0 & 1\end{pmatrix}$. Applying these formulae to Equation 22, we know that

$$(23)\qquad \begin{aligned} &\int_W \phi\eta q^r d\varphi_b = S(1-b^m\psi\overline{\xi}(b))a(p,f_0)^{\mu-\beta-\lambda}G(\omega_0)\langle h, f_0\rangle^{-1} \\ &\cdot\sum_{0<t|N_1} \mu(t)\omega_0(t)t^{-1}\langle f_0[\tau]_k, (\theta(\phi\eta)[\tau]_{\kappa+\alpha})J_{k-\kappa-\alpha,N_0}(t^{-1}N_1 z, -r, \overline{\omega}_0)\rangle_{Mp^\beta} \end{aligned}$$

where

$$S = \pi^{-m-r}2^{-m-2r}(\sqrt{-1})^{2k-m}M^{m/2}p^{m\beta/2+(\beta+\lambda-\mu)(k/2-1)N_0^{-1}\Gamma(m+r)}.$$

Then, the evaluation 17 follows from the formula given in [**Shi77**, p. 217].

1. A sketch of the Proof of Theorem 5.1

Finally we give a sketch of the proof of Theorem 5.1. We keep the same notation as in the statement of Theorem 5.1.

$$g = \sum_{n=0}^{\infty} b(n) e^{2\pi i n z}$$

be the fixed modular form in $\mathcal{M}_l(\Gamma_0(N), \omega)$ with $b(n) \in \overline{\mathbb{Q}}$. Let n be a positive integer and ϕ be an arbitrary function on $Y_n = \mathbb{Z}/Np^n\mathbb{Z}$ with values in $\mathbb{C}$. Define

$$g(\phi) = \sum_{n=0}^{\infty} \phi(n) b(n) e^{2\pi i n z},$$

as a function on $\mathfrak{H}$.

Proposition 1.1. *For any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N^2 p^{2v})$, we have the following transformation formula:*

$$g(\phi)[\gamma]_l = \omega(d) g(\phi_a),$$

where ϕ_a is a function on $\mathbb{Z}/Np^n\mathbb{Z}$ defined by

$$\phi_a(y) = \phi(a^{-2}y).$$

Proof. See Proposition 8.1 [**Hid85**]. ■

Now we shall give a sketch of a proof of Theorem 5.1. Fix an integer $b > 1$ that is prime to Np, and let r and m be integers with $r \geq 0$ and $m > 0$. Define the Eisenstein series $E^b_{m,n}(a)$ for each $a \in Y_n^\times$ by 9 with N in place of M. Write $N = N'p^\lambda$ with an integer N' that is prime to p and let ψ' be a Dirichlet character modulo Np^u for some $u > 1$. Define, for each $y \in Y_n$ (with $n \geq u$),

$$\Phi_n(y) = \Phi_n(y; r, m, \psi') = \sum_{a \in Y_n^\times} \psi'(a) H[g_n(a^2 y) \delta^r_m E^b_{m,n}(a)]. \tag{24}$$

Then, the system $\{\Phi_n(y)\}$ defines a bounded measure $\Phi(r, m, \psi')$ with values in $\overline{\mathcal{M}}_{l+m+2r}(NN', \psi'\omega; K)$ for a suitable finite extension K of $\mathbb{Q}_p$. We now define, the ordinary part $\Phi^O(r, m, \psi')$ of the measure $\Phi(r, m, \psi')$ by

$$\int_Y \phi d\Phi^O(r, m, \psi') = e[\int_Y \phi d\Phi(r, m, \psi').] \tag{25}$$

Then, the measure $\Phi^O(r, m, \psi')$ has values in the finite dimensional vector space $\mathcal{M}_{l+m+2r}(\Gamma_0(NN'p^u), \psi'\omega; K)$ by Theorem 3.2. Let k be an integer with $k > l$. If r is an integer with $0 \leq 2r < k - l$, we have, for any $\phi \in \mathcal{C}(Y; K)$,

$$\int_Y \phi d\Phi^O(r, k-l-2r, \psi') = (-1)^r \int_Y \phi(y) y_p^r d\Phi^{\mathrm{O}}(y; 0, k-l, \psi'), \tag{26}$$

where y_p is the projection of $y \in Y = \mathbb{Z}/N'\mathbb{Z} \times \mathbb{Z}_p$ to the factor $\mathbb{Z}_p$. Let f be a primitive form of weight $k > l$, of conductor C, and with character ψ. Assume that $\left|a(p, f)\right|_p = 1$ and that K contains all the Fourier coefficients of f. Let f_0 be the ordinary form associated with f and let C_0 be the smallest level of f_0. Write $C_0 = C'p^\mu$ with an integer C' prime to p and assume that $C' \mid N'$. Then, the measure $\Phi^O = \Phi^{\mathrm{O}}(0, k-l, \psi\overline{\omega})$ has values in the space $\mathcal{M}_k(\Gamma_0(N'^2 p^\mu), \psi; K)$. Let Tr denote the trace operator of $\mathcal{M}_k(\Gamma_0(N'^2 p^u), \psi; K)$ onto $\mathcal{M}_k(\Gamma_0(C_0), \psi; K)$. Then, the bounded measure φ_b on Y in Theorem 5.1 can be defined by

$$\int_Y \phi d\varphi_b = l_f[\mathrm{Tr}\left(\int_Y \phi d\Phi^{\mathrm{O}}\right)], \tag{27}$$

where l_f is the linear form on $\overline{\mathcal{M}}_k(C_0, \psi; K)$ attached to f. The evaluation of the integral $\int_Y \phi(y) y_p^r d\varphi_b(y)$ for any locally constant function ϕ with

$$\phi(zy) = \chi(z)\phi(y) \quad \text{for all } z \in Y^\times \text{ and } y \in Y.$$

can be carried out in exactly the same fashion as done above for the case of theta series.

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